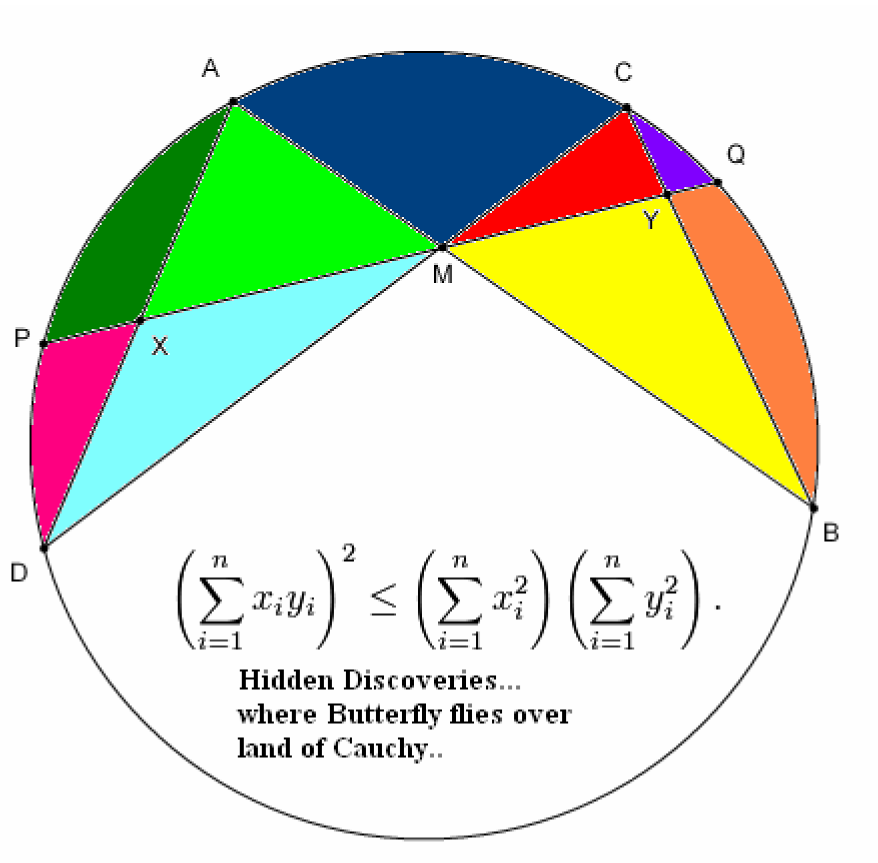

HIDDEN DISCOVERIES OPEN MATHEMATICS CHALLENGE

SOLUTIONS

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1. (001) We can solve this by modular arithmetic:

$$\sum_{n=1}^{4} n^5 = 1^5 + 2^5 + 3^5 + 4^5 \equiv 1+2+3+4 \equiv 5 \pmod 5$$
$$\sum_{n=1}^{4} n^4 = 1^4 + 2^4 + 3^4 + 4^4 \equiv 1+1+1+1 \equiv 4 \pmod 5$$
$$\sum_{n=1}^{4} n^3 = 1^3 + 2^3 + 3^3 + 4^3 \equiv 1+3+2+4 \equiv 5 \pmod 5$$
$$\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2 \equiv 1+4+4+1 \equiv 5 \pmod 5$$
$$\sum_{n=1}^{4} n = 1+2+3+4 \equiv 5 \pmod 5$$

So, $\sum_{n=1}^{4} (n^4 + n^3 + n^2 + n) \equiv 4 \pmod 5$. Then, $\sum_{n=1}^{4} n^5 - 4 \equiv 1 \pmod 5$. Thus, 1 is the remainder.

<u>OR</u>

We can brute force and do this by hand.

$$K = (1^5 + 2^5 + 3^5 + 4^5) - (1^4 + 2^4 + 3^4 + 4^4) - (1^3 + 2^3 + 3^3 + 4^3)$$
$$-(1^2 + 2^2 + 3^2 + 4^2) - (1+2+3+4)$$
$$= 1300 - 354 - 100 - 30 - 10 = 806$$

1 is the remainder when 806 is divided by 5.

<u>OR</u>

We could guess… It's either 0,1,2,3, or 4. ☺

2. (048)

$$\sqrt{3+2\sqrt{2}} = a+b \to 3+2\sqrt{2} = a^2 + 2ab + b^2$$
$$a^2 + b^2 = 3, 2ab = 2\sqrt{2} \to a = 1, b = \sqrt{2}$$
$$\therefore \sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$$
$$\sqrt[2010]{(1-i)^{2008} + (1+i)^{2008}}$$

$$(1-i)^{2008} = \left(\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)^{2008} = (\sqrt{2})^{2008}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{2008} =$$
$$2^{1004}\left(\cos\frac{2008\pi}{4} - i\sin\frac{2008\pi}{4}\right) = 2^{1004}$$

Note that since $\sin\frac{2008\pi}{4} = 0, (1+i)^{2008}$ is also 2^{1004}.

$\sqrt[2010]{(1-i)^{2008}+(1+i)^{2008}} = \sqrt[2010]{2^{1005}} = 2^{\frac{1005}{2010}} = 2^{\frac{1}{2}} = \sqrt{2}$

So, $z = 1$.

Hence, it is given that $P(1)$ and $P(0)$ are both odd. For any even integer $2k, P(2k)$ is odd (why? $P(0)$ being odd hints $a_0 =$ odd). Similar logic shows that $P(2n+1)$ will be in same parity and because $P(1)$ is odd, $P(2n+1)$ is odd. Combining those two, $P(x)$ is odd for all x. To have integer roots, $P(x)$ must equal zero, but zero is EVEN. So, $P(x)$ has no integer roots and $Q = 0$. Thus:

$$\text{Answer} = (0+3)(0+4)^2 = (3)(16) = 48$$

OR

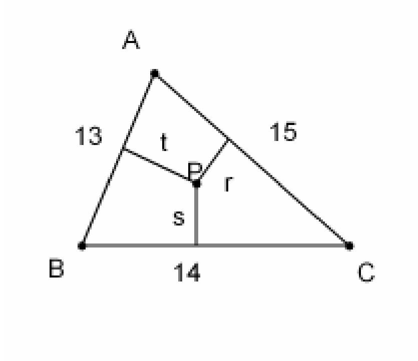
Without using DeMoivre's Theorem, note that:

$(1-i)^8 = 16 = 2^4$

So, $(1-i)^{2008} = ((1-i)^8)^{251} = (2^4)^{251} = 2^{1004}$.

$(1+i)^8 = 16 = 2^4$ and same logic follows.

3. (351)



$[ABC] = 84$ (by using Heron's Theorem or memorization)

Note that the area is also:

$$[ABC] = \frac{1}{2}(13t + 14s + 15r)$$

By Cauchy-Schwarz Inequality:

$(13t + 14s + 15r) \leq (13^2 + 14^2 + 15^2)(t^2 + s^2 + r^2)$

$13t + 14s + 15r = 84 \cdot 2 = 168$

$13^2 + 14^2 + 15^2 = 169 + 196 + 225 = 590$

$t^2 + s^2 + r^2 = \frac{84}{295} = \frac{168}{590}$

So, above inequality is actually an equation. This means that $\frac{13}{t}=\frac{14}{s}=\frac{15}{r}$. So:

$$t^2+s^2+r^2=\left(\frac{13s}{14}\right)^2+(s)^2+\left(\frac{15s}{14}\right)^2=\frac{168}{590}$$
$$\frac{169s^2+196s^2+225s^2}{196}=\frac{168}{590}$$
$$590^2s^2=168\cdot 196$$
$$s^2=\frac{168\cdot 196}{590^2}$$
$$s=\frac{2\sqrt{42}\cdot 14}{590}=\frac{14\sqrt{42}}{295}$$

So, $\frac{x\sqrt{y}}{z}=\frac{14\sqrt{42}}{295}$ and $x+y+z=14+42+295=351$.

4. (555) Seeing fraction, we think of $f(1-x)$.

$$f(1-x)+f(x)=\frac{4^{1-x}}{4^{1-x}+2}+\frac{4^x}{4^x+2}=\frac{4}{4+2\cdot 4^x}+\frac{4^x}{4^x+2}$$
$$=\frac{2\cdot 2}{2(2+4^x)}+\frac{4^x}{4^x+2}=\frac{2}{4^x+2}+\frac{4^x}{4^x+2}=1$$

So, we can match fractions in the S_n. Note that if n is even, there is one odd fraction in the form $f(\frac{1}{2})$ among all the paired ones. So, S_n is integer iff n is odd.

Note that:

(sum of odd integers from 333 to 777) = (sum of odd integers from 1 to 777) – (sum of odd integers from 1 to 331)

In math, it is:

$$=389^2-166^2$$

which comes from the identity (sum of n first odd positive integers is n^2). There are 389-166 odd integers in the interval given. So:

$$T=\frac{389^2-166^2}{389-166}=\frac{(389+166)(389-166)}{(389-166)}=555$$

5. (003)

The hardest part in this problem is to factor the expression:

$$ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)=(b-c)(a-b)(a-c)(a+b+c)=12$$

The left hand side actually appeared in 2006 IMO question but the actual problem involved much more difficult inequality and longer steps.

Now, 12 has 6 divisors: 1,2,3,4,6,12. Also, note that $a+b+c\geq 3$ since $a,b,c>0$.

Here comes the case-works:

Case I: $a+b+c=3$ then $a+b+c=3$ but that means $(b-c)(a-b)(a-c)=0$ so no solution.

From here on, we'll add more equation to ease our case-work:

$(a-b)+(b-c)=(a-c)$

Note that all three terms are products of the factored equation as well.

Case II: $a+b+c=4$ then $(b-c)(a-b)(a-c)=3$. So, the possible order of $(b-c),(a-b),$ and $(a-c)$ is $(1,1,3),(-1,-1,3),$ and $(1,-1,-3)$ and so on. But it is clear that none can satisfy $(a-b)+(b-c)=(a-c)$ so no solution.

Case III: $a+b+c=6$ then $(b-c)(a-b)(a-c)=2$.

Subcase I: $a-b=1, b-c=1, a-c=2$ $\rightarrow a=3, b=2, c=1$

Subcase II: $a-b=-2, b-c=1, a-c=-1$ $\rightarrow a=1, b=3, c=2$

Subcase III: $a-b=1, b-c=-2, a-c=-1$ $\rightarrow a=2, b=1, c=3$

Case IV: $a+b+c=12$ then $(b-c)(a-b)(a-c)=1$

So the possible orders for three products are $(1,1,1),(-1,-1,1),(-1,1,-1),$ and $(1,-1,-1)$. But again none satisfies $(a-b)+(b-c)=(a-c)$ so no solution.

$\therefore$ There are total of 3 (a,b,c) ordered triples, namely $(1,3,2),(2,1,3),$ and $(3,2,1)$.

6. (263)

The problem is best solved by considering for general case of n-sided regular polygon and k colors with $k \geq 3$. Let the number of ways to color regions to have adjacent ones differently be $P_{n,k}$. So, we look for $P_{8,3}$.

If Sam starts from OA_1A_2, he can accomplish this in k ways and $k-1$ ways for regions after this. But, there is an exception --- OA_nA_1. If you look at it as 8 regions, there are $k-2$ different ways. But when considered OA_nA_2 as a whole, this can be done $k-1$ ways. So, there is a bijection between two perspectives. Thus:

$P_{n,k}=k(k-1)^{n-1}-P_{n-1,k}$

with another exception $P_{3,k}=k(k-1)(k-2)$ (make sure you understand why this is true)

So:

$P_{n,k}=k(k-1)^{n-1}-k(k-1)^{n-2}+k(k-1)^{n-3}\cdots+(-1)^{n-4}k(k-1)^3+(-1)^{n-3}k(k-1)(k-2)$

Exception of the last term, the terms form geometric sequence after factoring out k.

$$=k\cdot\frac{(k-1)^n+(-1)^{n-4}(k-1)^3}{1+(k-1)}+(-1)^{n-3}k(k-1)(k-2)$$

$(-1)^{n-4}=(-1)^n$ and $(-1)^{n-3}=(-1)^{n-1}$ (make sure you understand this)

$= (k-1)^n + (-1)^n (k-1)^3 + (-1)^{n-1} k(k-1)(k-2)$
$= (k-1)^n + (-1)^n (k-1)[(k-1)^2 - k(k-2)]$
$= (k-1)^n + (-1)^n (k-1)$

So:
$(2)^8 + (-1)^8 (7) = 256 + 7 = 263$ ways.

Source: The original problem and its solution involving regular dodecagon came from *102 Combinatorial Problems*.

7. (647)

You can brute-force. Surprisingly, this is actually the easiest method although the calculation can be little bit tiring (but it's not as bad as it looks!)

$$P_{10}(2) = \binom{10}{2} + \binom{10}{5}(2) + \binom{10}{8}(2)^2$$
$$= 45 + 504 + 180 = 729$$

$$P_9(2) = \binom{9}{2} + \binom{9}{5}(2) + \binom{9}{8}(2)^2$$
$$= 36 + 252 + 36 = 324$$

$$P_8(2) = \binom{8}{2} + \binom{8}{5}(2) + \binom{8}{8}(2)^2$$
$$= 28 + 112 + 4 = 144$$

$\therefore$ Answer is $3(729) - 3(324) + 3(144) = 3(549) = 1647$ and the remainder is 647.

<u>OR</u>

The original problem of it went like this:
Prove that for above polynomial, following is true
$P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$
This came from Bulgaria Mathematics Olympiad. So, according to this original version, the answer is indeed $P_{11}(2)$. So, I'll prove the original problem to show why this problem is such a beautiful problem. For AIME convenience, I made the numbers nicer so it was easier to brute-force (I tend to think that there should be more than one ways to approach the problem --- methodically and extremely).

Now, to prove this, in general, all I need to consider is the coefficients of x^m where $0 \le m \le \left\lfloor \frac{n+1}{3} \right\rfloor$. Note that we are allowed to do this since for all of them, polynomial has same x (i.e. there is no term that goes like $P_n(x+1), P_n(x+5)$, etc…). So, it is enough if we can show below:

$$\binom{n+3}{3m+2} = 3\binom{n+2}{3m+2} - 3\binom{n+1}{3m+2} + \binom{n}{3m+2} + \binom{n}{3m-1}$$

Seeing something like this, we think of Pascal's Identity, which goes like this:

$$\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1}$$

So:

$$\left(\binom{n+3}{3m+2} - \binom{n+2}{3m+2}\right) - 2\left(\binom{n+2}{3m+2} - \binom{n+1}{3m+2}\right) + \left(\binom{n+1}{3m+2} - \binom{n}{3m+2}\right) - \binom{n}{3m-1}$$

$$= \binom{n+2}{3m+1} - 2\binom{n+1}{3m+1} + \binom{n}{3m+1} - \binom{n}{3m-1}$$

$$= \left(\binom{n+2}{3m+1} - \binom{n+1}{3m+1}\right) - \left(\binom{n+1}{3m+1} - \binom{n}{3m+1}\right) - \binom{n}{3m-1}$$

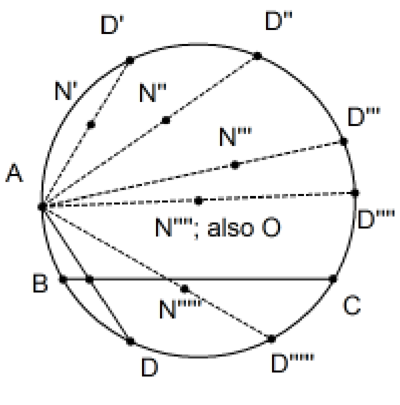
$$= \binom{n+1}{3m} - \binom{n}{3m} - \binom{n}{3m-1} = 0$$

and we are done. This is a phenomenal problem that shows how a simple combinatorial identity and simplification can save you tremendous amount of time in AIME (AIME, I think, is all about accuracy and time, and brute-force is neither although this problem wasn't too bad due to my courtesy).

8. (029)

This is a remake of old AIME 1983 #15, but with different diagram and length. So unless the contestant memorized the problem's wording, it was pretty difficult to recall that this was indeed about the same problem.

Let N be the midpoint of $\overline{AD}$. The locus of circle with A and N is the circle with P with radius 10. To find the radius, it is necessary to have an excellent thinking or a program. Below shows why this locus extends as far as radius of 10:



Thus, because the distance of AN remains constant and we know AN'''' is half of the radius of the circle with center O, this locus is used. Now, this circle can intersect BC in no points, one point, or two points. By problem, there is only one point so that's at N. Let M be the midpoint of $\overline{BC}$. Thus, $BM = MC = 12$.

So, $OM = \sqrt{20^2 - 12^2} = 16$ by applying Pythagorean Theorem.

Then, $\tan \angle BOM = \frac{12}{16} = \frac{3}{4}$

Now, M is the midpoint of segment BC and P is the center from the first construction. Note that $PN \perp BC$, so construct point K in OM such that $PK \perp OM$. Then:

$OM = PN + OK$
$= PN + PO \cos \angle AOM$
$= r(1 + \cos \angle AOM)$
$16 = 10(1 + \cos \angle AOM)$
$\cos \angle AOM = \frac{3}{5}$
$\rightarrow \sin \angle AOM = \frac{4}{5}$
$\tan \angle AOM = \frac{4}{3}$

Note that $\angle AOB = \angle AOM - \angle BOM$.

$\tan \angle AOB = \tan(\angle AOM - \angle BOM)$
$= \frac{\tan \angle AOM - \tan \angle BOM}{1 + \tan \angle AOM \tan \angle BOM} = \frac{\frac{4}{3} - \frac{3}{4}}{1 + \frac{4}{3} \cdot \frac{3}{4}} = \frac{7}{24}$

So, $\cos \angle AOB = \frac{24}{25} = \frac{24}{5^2}$ and answer is $24 + 5 = 29$.

9. (400)

Note that m is simplified as:

$$\sum_{i=0}^{44} \frac{\sin^2(2i+1)}{\cos^2(2i+1)} = \tan^2(2i+1) = \tan^2 1^\circ + \tan^2 3^\circ + \cdots + \tan^2 89^\circ$$

This part (the tangent square sum) is the problem that appeared in Hong Kong's Mathematics Magazine called *Mathematics Excalibur*. Note that $\cos 90^\circ = 0$. By DeMoivre's Theorem, we can establish:

$\cos 90\theta + i \sin 90\theta = (\cos\theta + i \sin\theta)^{90}$

Note that $i^2 = -1$ and $i^4 = 1$ so using Binomial Theorem and expanding and looking at the only real part:

$$0 = \sum_{k=0}^{45} (-1)^k \binom{90}{2k} \cos^{90-2k}\theta \sin^{2k}\theta$$

Now, divide $\cos^{90}\theta$ in both sides:

$$0=\sum_{k=0}^{45}(-1)^k\binom{90}{2k}\frac{\sin^{2k}\theta}{\cos^{2k}\theta}$$

$$0=\sum_{k=0}^{45}(-1)^k\binom{90}{2k}\tan^{2k}\theta$$

Let $\tan^2\theta=x$ then:

$$0=\sum_{k=0}^{45}(-1)^k\binom{90}{2k}x^k$$

So, the sum of the roots of this polynomial is also the sum in the problem. That is $\binom{90}{88}=4005$. Hence, $\lfloor\frac{m}{10}\rfloor=\lfloor\frac{4005}{10}\rfloor=400$.

10. (557)

Clearly, probability can be found by considering the quotient between selected outcomes' chances and total chances. Total chances for n is 3^n. For selected outcome, let's think backward.

$n=1\rightarrow 0$

$n=2\rightarrow 1$ (why? 1 bridge to come back from the island that Eric just moved to)

$n=3\rightarrow 3\cdot(3-1)$ (why? You could be coming from one of three islands and must have chosen combinations other than ones from $n=2$)

$n=4\rightarrow 3\cdot(3^2-(3-1))$

$n=5\rightarrow 3\cdot(3^3-3^2+(3-1))$

$n=6\rightarrow 3\cdot(3^4-3^3+3^2-(3-1))$

$n=7\rightarrow 3\cdot(3^5-3^4+3^3-3^2+(3-1))$

$n=8\rightarrow 3\cdot(3^6-3^5+3^4-3^3+3^2-(3-1))$

So, the probability is:

$$\frac{3\cdot(3^6-3^5+3^4-3^3+3^2-(3-1))}{3^8}=\frac{547}{3^7}$$

$a+b+c=547+3+7=557$

<u>OR</u>

Using the logic above, we can establish a recursive function.

$$f(n+1)=\left(\frac{1}{3}\right)(1-f(n))$$

where $f(k)$ is the probability for k km.
So:
$f(1) = 0$
$f(2) = \frac{1}{3}$
$f(3) = \frac{2}{9}$
$f(4) = \frac{7}{27}$
$f(5) = \frac{20}{81}$
$f(6) = \frac{61}{243}$
$f(7) = \frac{182}{729}$
$f(8) = \frac{547}{2187} = \frac{547}{3^7}$

OR

This problem, when considered as 3D, becomes a regular tetrahedron. It was 1985 AIME #12. In one of the solutions I saw, there was a very interesting point made by user ComplexZeta (Simon Rubinstein-Salzedo). He said that after a while, probability for one to be at one vertex is about $\frac{1}{4}$. So, even in this problem, $\frac{2187}{4} = 546.75$, which rounds to 547. This is a quick solution although it's not rigorous.

$\lim_{n\to\infty} f(n) = \frac{1}{4}$ (it's clear even without use of calculus)

11. (005)
$z^{2009} = 1$ so 2009th roots of unity are the solutions to this equation, and are in the form $\cos\left(\frac{2\pi k}{2009}\right) + i\sin\left(\frac{2\pi k}{2009}\right)$ for $k = 0, 1, \cdots, 2008$. Without loss of generality, let $v = 1$ then:

$$|v+w|^2 = |\cos\left(\frac{2\pi k}{2009}\right) + i\sin\left(\frac{2\pi k}{2009}\right) + 1|^2$$

$$= \left|\left[\cos\left(\frac{2\pi k}{2009}\right) + 1\right] + i\sin\left(\frac{2\pi k}{2009}\right)\right]^2$$

$$= \cos^2\left(\frac{2\pi k}{2009}\right) + 2\cos\left(\frac{2\pi k}{2009}\right) + 1 + \sin^2\left(\frac{2\pi k}{2009}\right)$$

$$= 2 + 2\cos\left(\frac{2\pi k}{2009}\right)$$

Since:

$|v+w| \geq \sqrt{2+\sqrt{2}}$
$|v+w|^2 \geq 2+\sqrt{2}$
$2+2\cos\left(\frac{2\pi k}{2009}\right) \geq 2+\sqrt{2}$
$\cos\left(\frac{2\pi k}{2009}\right) \geq \frac{\sqrt{2}}{2}$
$-\frac{\pi}{4} \leq \frac{2\pi k}{2009} \leq \frac{\pi}{4}$

So, $k = 251, 250, \ldots - 250, -251$ without including 0 since $v \neq w$. So, there are 502 values of k that'll work.

So, $\frac{m}{n} = \frac{502}{2008} = \frac{1}{4}$ and thus, $m+n = 1+4 = 5$.

*The original problem with $z^{1997} - 1 = 0$ and $|v+w| \leq \sqrt{2+\sqrt{3}}$ was in AIME 1997 #14.

12. (701)

Clearly, $x = -1$ is the root to the cubic. So, it factors as

$\left(x - \frac{\sqrt{145}}{2}\right)\left(x + \frac{\sqrt{145}}{2}\right)(x+1) = 0$

Hence, $x = \frac{\sqrt{145}}{2}$.

Now, the key to this part is to understand the location of D. Note that for triangle with circumradius R, inradius r, and distance d between incenter and circumcenter, we have this identity:

$$d^2 = R^2 - 2rR$$

Also, note that 20-21-29 is a right triangle so:

$[ABC]_{area} = (20)(21)\left(\frac{1}{2}\right) = 210$

$[ABC]_{area} = rs = \frac{abc}{4R}$

From these two, $r = 6$ and $R = \frac{29}{2}$. Plugging these into the identity:

$d^2 = \left(\frac{29}{2}\right)^2 - 2(6)\left(\frac{29}{2}\right)$

$$d^2 = \frac{145}{4}$$
$$d = \frac{\sqrt{145}}{2}$$

So, this implies that D is indeed the incenter of triangle ABC. Therefore, the positive difference between two circles' areas is:

$$\pi\left(\left(\frac{29}{2}\right)^2 - 6^2\right) = \frac{697}{4}\pi$$

Hence, $k + q = 697 + 4 = 701$.

13. (711)

This problem is from Korea Mathematics Competition. I just added the denominator to make it an AIME problem (the top part, including about the ways to find it, is basically the original problem).

We can consider 20 tosses as spots. Note that the total outcomes then becomes 2^{20}. For the 6 tails, we can achieve it in $\binom{20}{6}$ ways. But, to fulfill "no more than 5 heads in a row" restriction, we must use Inclusion-Exclusion Principle.

$$\binom{20}{6} - (\text{1 spot with 5 or more heads}) + (\text{2 spots with 5 or more heads}) - \cdots$$

Now, $(\text{1 spot}) = \binom{7}{1}\binom{15}{6}$ since we must leave 5 heads for one spot (to be chosen from 7 spots around the 6 tails) and then choose 6 tails from remaining total 15 spots.

Similarly, note that $(\text{2 spots}) = \binom{7}{2}\binom{10}{6}$.

3 or more spots with 5 become irrelevant because $\binom{5}{6} = 0$.

Thus, the answer is:

$$\frac{m}{n^2} = \frac{\binom{20}{6} - \binom{7}{1}\binom{15}{6} + \binom{7}{2}\binom{10}{6}}{2^{20}} = \frac{8135}{1024^2}$$

So, $\lfloor\frac{m-n}{10}\rfloor = \lfloor\frac{8135-1024}{10}\rfloor = 711$.

*For Principle of Inclusion-Exclusion, I have taken the image off from Mathworld to define it. This image is not mine because it belongs to Mathworld.

$$|A_1 \cup A_2 \cup \ldots \cup A_p| =$$
$$\sum_{1\le i\le p} |A_i| - \sum_{1\le i_1<i_2\le p} |A_{i_1} \cap A_{i_2}| + \sum_{1\le i_1<i_2<i_3\le p} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \ldots + (-1)^{p-1} |A_1 \cap A_2 \cap \cdots \cap A_p|.$$

14. (126)

Point P is called Brocard Point, and satisfies:

$\cot\omega = \cot A + \cot B + \cot C$

where $\omega = \angle BCK = \angle ABK = \angle CAK$.

Using Law of Cosines to find $\cos A, \cos B$,, and $\cos C$, we can use Pythagorean Identity to find $\sin A, \sin B$, and $\sin C$.

$\cot A = \frac{2}{3\sqrt{5}}$

$\cot B = \frac{11}{8\sqrt{5}}$

$\cot C = \frac{2}{\sqrt{5}}$

So, $\cot\omega = \frac{97}{24\sqrt{5}}$ and $\tan\omega = \frac{24\sqrt{5}}{97}$. The answer is 126.

The idea (and basically the problem except the lengths are different) came from 1999 AIME #14.

**The proof of Brocard Point is left to readers to find. I tried to find it on the Internet but was pretty unsuccessful though. ☹ **

15. (000)

This is a tough problem. It's tough because there are lots of things that you have to notice immediately or you will spend very large time doing nothing useful. There is nothing difficult about the problem other than some calculations.

Let's analyze *f(x)*:

By coefficient of $\cos^3 x$, we can guess that this has to do with triple-sum angle formula which goes like this:

- $\cos(3y) = 4\cos^3(y) - 3\cos(y)$

Similarly but more commonly, we know

- $\cos(2y) = \cos^2(y) - \sin^2(y) = 2\cos^2(x) - 1$

Adding these two, we get:

$$= 4\cos^3 x + 2\cos^2(x) - 3\cos(x) - 1$$

Up to here is probably the one of most difficult part in this problem. Now, comparing this with the problem's equation, we see that we just have to add $\cos^2(x)$ term and +2. So,

$f(x) = \cos(3x) + \cos(2x) + \cos^2(x) + 2$

Based on this, we can try to solve the equation… not! The given result is very ugly but there are some parts that interest us: √2 and √3. Although there is no easy to go about this, we can guess choices like 30, 45, and so on. None of the common values work. So, let's think about this in little bit more difficult way: 15.

$$\cos(3\times15) = \cos(45) = \frac{\sqrt{2}}{2}$$

$$\cos(2\times15) = \cos(30) = \frac{\sqrt{3}}{2}$$

$$\cos^2(15) = \cos^2(45-30) = \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2$$

$$= \frac{2+\sqrt{3}}{4}$$

$$\therefore \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + \frac{2+\sqrt{3}}{4} + 2$$

$$= \frac{2\sqrt{2}+2\sqrt{3}+2+\sqrt{3}+8}{4}$$

$$= \frac{2\sqrt{2}+3\sqrt{3}+10}{4}$$

So, x is 15 degrees. Now, to calculate $S_2(15)$, we can use Pythagorean Identity to find cotangent of 15 degree. But using the identity in the Appendix, we see that the result is simply 0. So, $4x = 4(0) = 0$. Done.

Appendix

Proof of Important Trigonometry Identity from Problem #15

Identity: For triangle ABC, there is an identity such that:

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

This is an important identity, and I highly recommend you know of it for USAMO and other higher levels of math competitions.

Proof:

Note that A+B+C = 180 so A/2+B/2+C/2 = 90 (in degrees).

Rewriting everything in terms of sine and cosine:

$$\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$=\frac{\cos\frac{A}{2}\sin\frac{B}{2}+\sin\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$=\frac{\sin\left(\frac{A}{2}+\frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

Using the equation very early in the proof, we can now apply sin(z) = cos(90-z) identity:

$$=\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$=\left(\cos\frac{C}{2}\right)\left(\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)$$

$$=\left(\cos\frac{C}{2}\right)\left(\frac{\sin\frac{C}{2}+\sin\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\right)$$

Now, using the same idea:

$$= \cot\frac{C}{2}\left(\frac{\cos\left(\frac{A}{2}+\frac{B}{2}\right)+\sin\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}\right)$$

$$= \cot\frac{C}{2}\left(\frac{\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}\right)$$

$$= \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

I highly recommend that you work out this on your own as well to see how this is true!