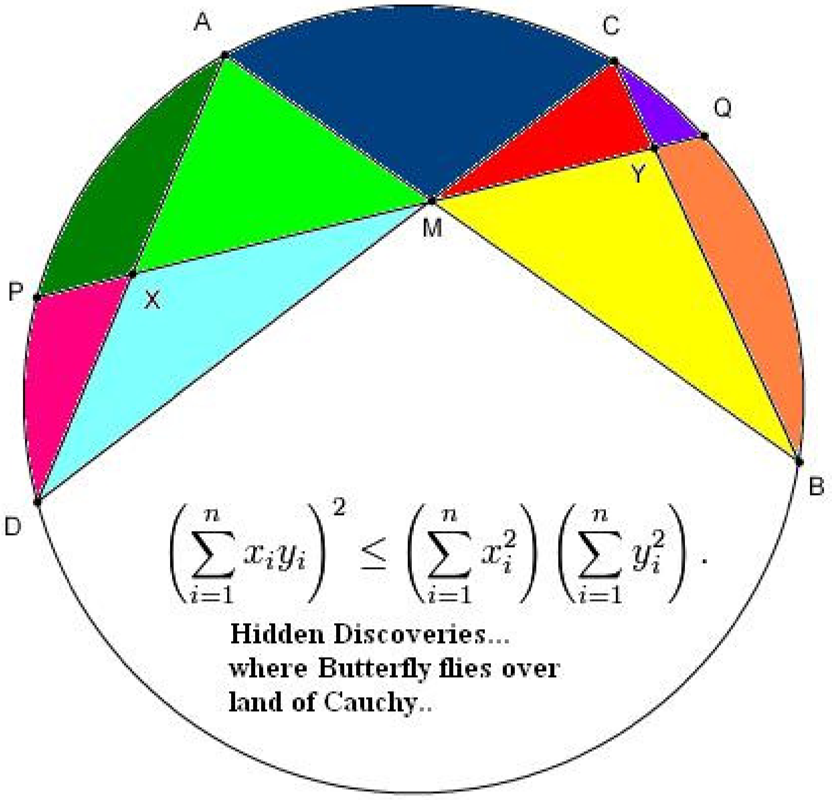



# Hidden Discoveries

## Intermediate Geometry Techniques

## First Edition

# I. INTRODUCTION

This document is created because at this point, there is no Intermediate Geometry by Art of Problem Solving. There is no such class or book although the talks of them have been mentioned for past couple years. So, in order to address that need, I have decided to create a helpful guide that can hopefully be a supplement to that text when it comes out.

Nevertheless, there are several books that are excellent for this.

1) *Art of Problem Solving Volume I* by Sandor Lehoczky and Richard Rusczyk

- This book, along with solution, has a very detailed geometry section that can be used for both AMC and AIME. The harder theorems like Ceva's and Ptolemy's are not mentioned though.

2) *Art of Problem Solving Volume II* by Sandor Lehoczky and Richard Rusczyk

- This book, a sequel to the first one, contains more trigonometric ways to solve geometry problems and explore much difficult geometric concepts, including construction. It touches briefly on Olympiad level but mostly as a supplement for AIME level geometry not found in the first volume.

3) *Challenging Problems in Geometry* by Alfred S. Posamentier and Charles T. Salkind

- This book is simply a collection of many problems. Some are AIME level while others are AMC level. It is certainly a good book to just practice but the problems are not always that of MAA. It is more straightforward and asks you to find exactly what they want (whereas MAA's problems can be wordy and make you to think more).

4) *Geometry Revisited* by H. S. M. Coxeter and S. L. Greitzer

- This book is rather opposite of the third one because it has brief number of problems and a large proof. Some theorems are very advanced, and the book is more oriented toward Olympiad than AIME. It's still intermediate in some ways but it definitely contains materials that are not for AIME.

# II. OTHER RESOURCES

I never really liked mass points but if you want to learn this method (which is pretty useful for ratio problems), you can get it here:

http://mathcircle.berkeley.edu/mpgeo.pdf

At this point, the link works but it may not for future so don't email me because of this.

You can also purchase geometric programs (Geometer's Sketchpad) if you are a poor drawer like myself. If you want to get a free one, you can get KSEG. This is a program I use because I don't have money to spend on geometry program.

http://www.mit.edu/~ibaran/kseg.html

It's still an excellent program. Geometry programs are nice that it helps you to visualize problems if you don't understand how they are constructed. ☺

## III. MAIN GUIDE

This document is first edition based on about 20-30 AIME geometry problems. Thus, it does not extend over for all AIME problems and is incomplete. That is why this guide is first edition, and when I have more time to do more AIME problems, I will create second edition and so on.

1. All of these rules only work by doing problems! I have found these rules by doing *actual* AIME problems! So these are not just ideas. They do work in real AIME problems!
2. When faced with angles (e.g. $x$, $2x$, …), try to construct a circle with some or all points to form something useful, like parallelogram.
3. When opposite angles of the quadrilateral add up to 180 degrees, the quadrilateral is cyclic, and it's very useful!
4. With coordinate geometry problems, remember that direct algebra, with correct calculation, will give the right answer but with very messy algebra work. If nothing works, think about this ugly way with systems of equations.
5. With coordinate geometry problems excluding direct algebra methods, it is good to make constructions that create isosceles triangle because isosceles triangle makes the process to solve problems regarding slopes very convenient (ex. The angle bisector of an isosceles triangle goes through opposite side's midpoint, and you can find slope by the vertex and this midpoint).
6. When faced with numerous similar triangles, one way to solve this type of problems is to list all the similar ratios and see if multiplying two or more ratios give desired fraction, or the fraction equivalent to it. This method is easier than just staring at triangles and guessing which ratios to use (which make the process not only harder but also longer!).
7. With problems involving trigonometric function, first see if common laws like Law

of Sine, Extended Law of Sine (this involves circumradius), Law of Cosine, and Law of Tangent can work.

8. Advanced trigonometric law that is quite useful is this one:

- tan A + tan B + tan C = tan A*tan B*tan C

9. With cevians and ratio problems, one of the most ways to approach problem is by area ratio. Remember that triangles sharing same height will have ratio by their bases so using this idea can simplify problem greatly! Also, it is important to combine the areas with same methods but from different starting points. This often gives a desired form to work with.

10. In the right triangle ABC with hypotenuse AB, the altitude CH, where H is on AB, has length of $\sqrt{(AH)(BH)}$. This is a very useful application, and it is recommended to see this by similar triangles.

11. With points in the coordinate plane, one method to find their relationship is through matrices.

For point (x,y) with rotation angle θ:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Thus, the new rotated point (x',y') has the coordinate of (x',y').

$x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$

12. Another point to remember is with complex number points, and rotation including not just point but that of lines:

With rotation of lines, use slope and y-intercept:

Slope: Recall that slope is tangent of angle. So, rotation of Θ with slope of line *m*:

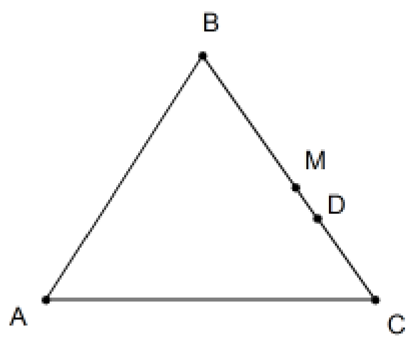
New slope = $\dfrac{m + \tan\theta}{1 - m \cdot \tan\theta}$

y-intercept: In polar coordinates, this point has coordinate ($r$,90°). So, point in line before the rotation has coordinate ($r$,90-Θ°).

Since the slope of line before rotation is $\tan(90^\circ - \theta)$, this line is $y = \tan(90^\circ - \theta)x$. By setting this with original equation, then (x,y) can be found. Using $r^2 = x^2 + y^2$, we can find the value of *r*. So, the new line has the form:

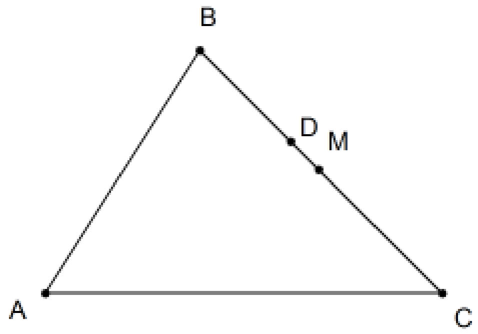
$$y = \frac{m + \tan\theta}{1 - m \cdot \tan\theta}x + r$$

With complex number rotation:
$x + yi$ by Θ degree is:
$(x + yi)(\cos\theta + i\sin\theta)$
(Thanks to user TZF for #11 and #12)
13. With earlier geometry problems, understand that MAA will not put extremely difficult theorem-needed problems. Instead, it will require a good *and* correct diagram with right math calculation. Good and correct diagram consists of two parts: using ALL of problem's description and drawing in a realistic way. ALL the information in the problem is mentioned by the problem writer because they are important. Do not assume (this is like cursing when it comes to geometry) that the problem is solvable with partial information. You will likely fall into writer's trap, or trap that you created yourself. Drawing in a realistic way is often mentioned a lot but rarely emphasized enough. Let me give you an example:
*In triangle ABC, M is the midpoint of BC. AB = 18, AC = 20, and <BAC is bisected by line AD where D is on BC….(the problem goes on)*
Based on this information, careless students create bad diagrams while cautious students make good diagrams. The distinction appears to be minor but is significant when it comes to problem solving.
Bad:




AB = AC in drawing (difficult to tell) so D is just placed "near" M. The student did not give much thought about this yet. As long as D is in BC, that is all student cared about. On the other hand, let's observe the case for good students.

Good:




AC > AB clearly so student used this fact to make another accurate construction of AD > BD (based on Angle Bisector Theorem) IN DIAGRAM. Thus, D is now placed above M instead of *below it or simply "near" it.*

This subtle difference is very important because with later problems, it can decide whether point is inside or outside the triangle.

14. What you should not do, however, is to be so caught up with correct diagrams that you take out rulers and spend time making *artistic* diagrams. I actually did this in AIME and although I managed to get the problem right, it took me about 15-20 minutes more than needed. I was so into making a big, accurate diagram that I incorrectly drew it, and erasing it took lots of time too. So, be realistic: you want to have an accurate diagram but don't spend time trying to make da Vinci-style geometry diagrams. Diagrams are not what count; the correct answers.

15. With right math calculations, one recommended way is to do calculations in two ways: by hands and by hands with little algebra. Example:

31 x 31

By hand:

```
  31
x 31
----
  31
 93
----
 961
```

By hand with little algebra:

Note that 31 = 30+1

So: 31 x 31 = (30+1)(30+1) = 900+60+1 = 961

This is what I mean by checking in to ways. There are many other ways, and I suggest each student finds the method that increases the accuracy to maximum.

16. Right triangles mean Pythagorean Theorem. Whether it's constant or algebraic expression, one method that you want to think about is Pythagorean Theorem with right triangles. This is often the most straightforward but it can be answers to problems, especially with easy geometry and keys to some harder problems.
17. With parallel lines, remember that these lines lead to numerous similar triangles, and more importantly, that if the similarity ratio of length is $x/y$, then the area ratio is $x^2/y^2$.

That's all for the first edition! When I get more free time and have solved more problems, I'll put up second edition with different tips. I understand this guide is most likely effective for intermediate geometry problems (#1-10) and that those after them are much more complex. But remember that several problems in geometry, even the harder ones, have basis in the simplest ways, and that often IS the right way to solve them.

I've learned a lot while making this guide so I'm not good at geometry either. But I've seen from other people who are brilliant, and these people treat geometry like *games*. Don't be afraid to see geometry problem. Have fun and if you stay calm (as stated in Problem Solving Strategy guide), you will have clear mind, carry out accurate math calculations, and see ways to solve geometric problems even if (most likely) your first method failed.

Good luck!

December 30, 2008
Ji Woong Park