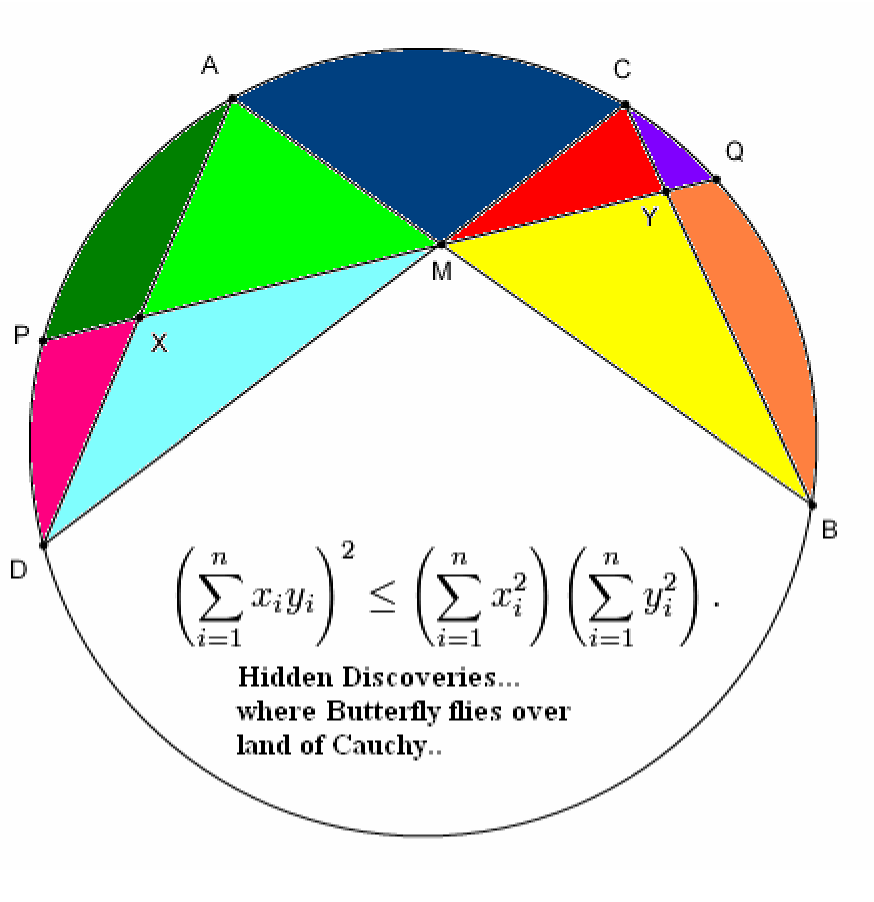

HIDDEN DISCOVERIES

NUMBER THEORY GUIDE

I. About This Guide

 All the problems in this guide are based on real AIME problems but all of them are **original**. Therefore, please give high respect to Hidden Discoveries for making the guide.

Problems are divided into two categories: Easy and Medium. My goal was to at first make three sections of difficulty but due to time span and other ongoing projects, I decide to pause at #16 on Medium.

At the end of the guides, you will find hints to the problems. Hints will probably be one or two sentences pointing out the key part of the questions. It is highly recommended that students take hints and try as long as they can before they peek a look at answers. Once they look at answers, they are recommended to get that answer by trying more and then eventually comparing with solutions.

There is no introductory problem or anything like this to guide. Students are required to try all the problems and use this as their individual problem set.

My hope is that teachers also use this guide as well. These problems are excellent for those who want to start math clubs or simply introduce an interesting aspect of number theory. For the terms that you do not understand, I highly recommend purchasing Art of Problem Solving books to self-study on your own for a little bit. I will not define each term in this guide because that is not the purpose of this guide.

Also, for the general problem solving simulation, I recommend that you take part in my competition – Hidden Discoveries Math Olympiad. The details can be found on the website (www.hiddendiscoveries.com).

II. Problems

Easy

1. Let Q be the product of first 100 positive odd integers. Compute the largest integer m such that Q is divisible by 5^m (Modified from 2006 AIME II #3).
2. The integer p is the smallest positive multiple of 35 such that every digit is either 6 or 5. Find p (Modified from 1984 AIME #2).
3. For how many values of v such that the least common multiple of 5^7, 7^5, and v is 175^5? (Modified from 1998 AIME #1)
4. Find the second smallest prime that is the fifth term of an increasing arithmetic sequence with all four preceding terms being prime as well (Modified from 1999 AIME #1).
5. Find the largest 2-digit prime factor of the integer $\binom{100}{50}$ (Modified from 1983 AIME #8).
6. Let u be the smallest positive integer that is a multiple of 72 and has exactly 72 positive integral divisors, including 1 and itself. Compute $u/72$ (Modified from 1990 AIME #5).
7. Let M be the *second* largest positive integer with the following property: reading from left to right, each pair of consecutive digits of M forms a perfect square. What is the product of digits of M? (Modified from 2001 AIME II #1)
8. How many positive integers have exactly 3 proper divisors, each of which is less than 30? (Modified from 2005 AIME I #3)
9. Suppose q is a positive integer and e is a single digit in base 10. Find q if

   $$\frac{d}{540} = 0.e07e07e07e07....$$

   (Modified from 1989 AIME #3)
10. Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is 1/37 of the original integer (Modified from 2006 AIME I #3).

Medium

11. What is the largest positive integer that is not the sum of a positive integral multiple of 36 and a positive composite integer? (Modified from 1995 AIME #10)
12. In a parlor game, the shaman asks one of the participants to think of a three digit number (pqr) where p,q, and r represent digits in base 10 in the order indicated. The shaman then asks this person to form the numbers (prq), (qpr), (qrp), (rpq), and (rqp), to add these five numbers, and to reveal their sum, S. If told the value of S, the shaman can identify the original number, (pqr). Play the role of the shaman and determine (pqr) if S = 2635 (Modified from 1986 AIME #10).
13. A sequence of numbers $x_1, x_2, x_3, \dots x_{100}$ has the property that, for every integer k between 1 and 100, inclusive, the number x_k is k less than the sum of the other 99 numbers. Given that $|x_{75}| = m/n$, where m and n are relatively prime positive integers, find $m+n$ (Modified from 2000 AIME I #10).
14. The numbers in the sequence 37, 40, 45, 52,... are of the form $a_n = 36+n^2$, where $n = 1,2,3,\dots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers (Modified from 1985 AIME #13).
15. Call a positive integer N a 7-10 *double* if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the *second* largest 7-10 double? (Modified from 2001 AIME I #8)
16. For $\{1,2,3,\dots,n\}$ and each of its non-empty subsets, an alternating sum is defined as follows. Arrange the number in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. For instance, the alternating sum for $\{1,2,3,6,9\}$ is 9-6+3-2+1 = 6 and for $\{5\}$ it is simply 5. Find the sum of all such alternating sums for $n = 6$ (Modified from 1983 AIME #13)

III. Hints

1. Use the fact that there are odd integers that are not divisible by 5 at all, divisible by 5, divisible by 25, and so on.
2. You do not know the divisibility rule for 35. But you do know that for its factors.
3. Use prime factorization.
4. You can try to just start listing but think about what type of number has to be the first number.
5. Think about this problem. Simple.
6. There is a rule that states for any given number n with prime factors $p_1, p_2, p_3, \ldots$ with exponents $e^1, e^2, e^3, \ldots.$, you can find the number of total divisors this way:

   $$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_n^{e_n}$$

   The total number of divisors is calculated by:

   $$\text{Total Number of Divisors} = (e_1 + 1)(e_2 + 1) \cdots (e_n + 1)$$
7. List two-digit perfect squares…and think.
8. #6's identity + combinatorics.
9. Think about what type of denominator can replace 540.
10. Is the answer 2-digit? 3-digit? How can we express *abcdef*… in terms of 10 and/or its powers?
11. To find if number is composite or prime, check to see if it has divisor by taking its square root and dividing by primes less than its root.
12. Consider S + pqr. Is there anyway to rewrite this?
13. This is actually more of an algebra problem but it is excellent introduction to future number theory problems where an application of summation comes handy. How can we write the sum of other 99 numbers in respect to the whole sum?
14. For any pair of positive integers p and q, can you think of any relationship in terms of divisor of $p - q$?
15. Play with the numbers and think whether you should progress from base 7 or from base 10. This decision should make the problem much more approachable.
16. Work in the small cases and see what is so special about the alternating sum. Then think and go with it. You do not need to prove in AIME, remember?

III. Answers

1. 025
2. 665
3. 006
4. 053
5. 097
6. 175
7. 648
8. 048
9. 220
10. 925
11. 175
12. 473
13. 624
14. 145
15. 312
16. 192

IV. Solutions

1. There are several ways to do this problem but all use the same idea. First of all, you want to find the boundary that you are interested in – this occurs with 195 (the 100^{th} odd positive integer is 199). Since 195 = 5 x 39, you are considering from numbers from 5 x 1 to 5 x 39 except the desired numbers should be in form 5 x 2n+1 because otherwise, we will have an even product in the range. So, using this idea, we can count how many. I did not use PIE (Principle of Inclusion-Exclusion) because I never really liked that method but you can still write the odd integers from 1 to 39 and just count how many 5's are there. I'll leave this to readers to figure out but it's pretty clear that *m* comes out to be 25. Another way is to consider from 1 to 11, 13 to 21, 23 to 31, etc.. This method may be easier for beginners but either way, you will get 25 as *m*.
2. To find if number is divisible by 35, you can do this by testing rules of 5 and 7 – two factors of 35. In order for number to divide by 5, it must end in 5 or 0. Since *p* can only use 5 or 6, it ends in 5. For divisibility rule for 7, the method is this: take the last digit of number, double it, subtract it from the remaining number, and number is divisible by 7 if the number from subtraction is. The first number in mind is 65. Obviously, 5 x 2 = 10, which is greater than 6 so this is not divisible by 35. The second number is 655 (we are slowly increasing here). 5 x 2 = 10 and 65 – 10 = 55, which is not divisible by 7. The third number is 665 (not 656 since 5 is on the units digit). 66 – 10 = 56, which indeed is divisible by 7. 665 is divisible by 35 (quotient of 19) so this is value of *p*.
3. 175^5 is same as $(5^2)^5$ [which also is 5^{10}] x 7^5. So, because there is already 7^5, v can have as small as 7^0 to large as 7^5. But since there isn't 5^{10} in other two products, v must be 5^{10} in terms of power of 5. So, power of 5 for v is fixed where there are 6 values for powers of 7. This is why there are 6 values of v.
4. The first number has to be 5. 2 is not one of the sequences, obviously. Thus, the five numbers can either rotate in units digit of 1,3,5,7,9 in some order or stay with same digit (if the common increase is 10). If 5 is the units digit of any number other than 5, it will not be prime. So, we have to start with 5 and just slowly change the second prime number. The first sequence (as used in the real exam) turns out to be 5,11,17,23,29. The second such sequence is 5,17,29,41,53.
5. The original AIME problem required you to see that the two-digit prime integer must appear twice in order to stay. But note that in this case, that is not the case for prime integers larger than 50. So, we simply pick 97, largest 2-digit prime.

6. The hint should've helped significantly in this problem. The equation is very useful and has appeared in many AMCs. I can think of two examples – 2005 AMC 12 B #21 and 1996 AHSME #29. I suggest that if you found this problem to be somewhat obscure, check out those problems first and come back to this one. It might make more sense that way. Using the prime factorization, we know that $u = 2^3 3^2 k$ where k is a positive factor. We know that k could contain 2 or 3 or only one of them or none of those two prime numbers. Using the formula I gave, we can see that # of divisors of $u = (4)(3)k$ if k does not contain either 2 or 3. Because 72 = 4*3*3*2 (I didn't factor out 4 because we need at least one number to be big as 4), we can see if k does not contain either 2 or 3. In this case, the smallest way would be to have $k = 5^2 7^1$ or 175. I'll leave to readers to see why this is indeed the smallest way to do this (think about what would've happen if you increased the power of 2? 3? 5? Etc..). This is a straightforward problem but you had to know ways to do this or you would've been guessing (and multiplying 72 by 175… would take a long time to come by).
7. Writing out the perfect squares for two-digit numbers, we see that there are only three numbers that go beyond two-digit (or less). They are 1649, 649, and 81649. Thus, 1649 is the second largest and the product of its digit is 1*6*4*9 = 216. BUT, remember that there is another case: 36. So second largest case is actually 3649 and the answer is 3*6*4*9 = 648. Thanks to user jhangil for pointing out this.

   The original problem asked to find the three leftmost digits of largest one, which would've then made the answer 816.
8. By proper divisors, we mean numbers that divide n excluding n. There are 10 primes less than 30 so with primes in form $p_1^1 p_2^1$, there are 10C2 = 45 ways to do this. For primes that are in form p_1^3, there are 3 since square of 7 is greater than 30. So, total of 45+3 = 48 ways.
9. Note that 0.e07e07e07e07… is same as e07/999 where e07 is a three digit positive. Now cross-multiply to get 540*e07 = 999q. Note that because 999 = 27*37 and 540 is not divisible by 37, e07 must be. From here, you can use algebra and modular arithmetic (I'll leave this part to the readers) to see that it works when e07 = 407 = 37*11. So, q is: 540*407/999 = 27*20*37*11/(37*27) = 220 as desired.
10. It should be obvious that there is no such 2-digit integer. So, we are dealing with 3-digit integer *abc*. Let's write this as power of 10:

    $abc = 100a + 10b + c$

It is given that $10b + c$ is 37^{th} of abc. So we have an equation:

$(100a + 10b + c)/37 = 10b + c$

$100a + 10b + c = 370b + 37c$

$100a = 360b + 36c$

$25a = 90b + 9c = 9(10b + c)$

Now here comes number theory. Because there is 9 in RHS, a must be divisible by 9. Well, since a is one digit, $a = 9$. In that case, $b = 2$ and $c = 5$ to get both sides equal. Thus, the number is 925.

BEFORE MOVING INTO THE MEDIUM SECTION, I HIGHLY RECOMMEND THAT EACH STUDENT REVIEWS THE ABOVE 10 PROBLEMS. IT'S MUCH BETTER TO UNDERSTAND THESE FUNDAMENTAL CONCEPTS INVOLVING MODULAR ARITHMETIC, COMBINATORICS, ALGEBRA, AND OTHER BRANCHES OF PROBLEM SOLVING. A GREAT PROBLEM SOLVER IS NOT SOMEONE WHO CAN JUST DO NUMBER THEORY PROBLEMS. A GREAT PROBLEM SOLVER IS SOMEONE WHO CAN APPLY THE OUTSIDE KNOWLEDGE AND INCORPORATE THEM INTO THE MORE ADVANCED PROBLEMS TO MASTER THE TRUE ART OF PROBLEM SOLVING SKILLS.

11. I hope you guys took the last paragraph to the heart because from here, we are facing some real challenging number theory problems now. The key part to notice is that any prime number greater than 36 can be reduced to the composite or another prime number by subtracting 36 from it. Similarly, we can satisfy the problem by starting with prime number less than 36 and continuing until we hit a largest prime number that adding 36 will make a composite number. This satisfy the problem because now, we have found the largest number such that it is sum of positive integral factor of 36 (since we are adding at least one 36 from prime less than 36) and another prime number. Stop here if you do not get the concept. It may help if you start going 36, 37, etc… and figure out the concept. Below shows the methodical way to find the solution. Make sure you understand how this works. To check if the resulted number was prime, use the method given in the Hints.

Starting Prime	+36 once	+36 twice	+ 36 for 3x	+36 for 4x
2	38 (not prime)			
3	39 (not prime)			
5	41	77 (not prime)		
7	43	79	115 (not prime)	
11	47 (not prime)			
13	49 (not prime)			
17	53	89	125 (not prime)	
19	55 (not prime)			
23	59	95 (not prime)		
29	65 (not prime)			
31	67	103	139	175 (not prime)

Note that this solution does not even mention modular arithmetic or anything like that. You do not necessarily need to (although it helps) understand complex number theory or use a clean set of methods for AIME. If you solve it in any way and get it right, that is all that matters. It's not USAMO.

12. Adding S and pqr gives 222(p+q+r) (make sure you understand WHY this is true). So:

S + pqr = 222(p+q+r)

2635 + pqr = 222(p+q+r)

Also, we know that 2635 is 193 mod 222. So, in order for this equation to be true, pqr must be 29 mod 222. Thus, our options can be calculated by 29+222k where k is a positive integer. So:

251, 473, 695, 917

From here, there are two ways to do this. We can use the equation above or just brute force.

Method 1: Using the equation:

Because 2635 = 11(222) + 193, p+q+r needs to be at least 12. So, let's use the digital sum (sum of the digits in the number; in here, p+q+r).

251 is 8 so this is out.

473 is 14 so this is good.

695 is 20 so this is good.

917 is 17 so this is good.

We can now plug in the numbers and the digital sum to see that it works for 473, left side is smaller than the right side for 695 and 917.

473+2635 = 3108 = 222(14) = 222(4+7+3)

Method 2: Brute force intelligently

We can directly estimate the value of S with the given numbers.

For 251, it's 215, 512, 521, 125, 152... Approximately, 200+500+500+100+100 = 1400 so too small.

For 473, it's 437, 734, 743, 347, 374... Approximately, 400+700+700+300+300 = 2400 so pretty close.

For 695, it's 659, 596, 569, 965, 956... Approximately, 600+600+500+900+900 = 3500 so too large.

For 917, it's 917, 179, 197, 719, 791... Approximately, 900+100+200+700+800 = 2700 so pretty close.

So, testing 473 and 917, we see that 473 is the answer. Note that this method of approximation does not give you the correct answer. It is just a ballpark of what the total sum might be (give and take 100-300) but for numbers like 251 and 695, we can be sure to eliminate them.

13. Let S be the sum of all 100 numbers. With mathematical notation, this appears like this:

$$\sum_{1}^{100} x_i = S$$

This is a really nice thing to do because now, using the problem’s definition, we know:

$$x_k = S - x_k - k$$
$$2x_k = S - k$$
$$x_k = \frac{S-k}{2}$$

So, this shows that:

$$x_1 = \frac{S-1}{2}, x_2 = \frac{S-2}{2}, x_3 = \frac{S-3}{2}, \cdots x_{100} = \frac{S-100}{2}$$

Adding these gives the sum of S in LHS and bunch of S's in the RHS. Now, you can find the S (which turns out to be 25*101/49), and that x_{75} is -575/49, so the absolute valued sum is 575+49 = 624. I abbreviated the algebraic calculation because that is not really the core of this guide. But this problem is really nice for more difficult number theory problems because there is a high chance that you will be faced with complex summation, and often times (but not always), using sigma and mathematical notation to rewrite things can simplify the problem. Again, not always but this is one of things that you can put in your "number theory toolbox."

14. While this problem seems out of ordinary to be placed in Medium difficulty (since its number is 13), I have a good reason. This problem is only difficult because many students do not have an exposure to the method called Euclid's algorithm. I actually came across this technique long time ago, back in middle school because of text *Mathematical Circles (Russian Experiences)* by Dmitri Fomin, Sergey Genkin, and Illia Itenberg. I highly recommend this text to anyone who is interested in math in general because this book is not really competition oriented. To certain aspects, yes, but this problem does include some interesting topics (i.e. graph theory, invariants) that are rarely addressed in many other texts for high school students in America.

    Having that said, what is Euclid's algorithm? It's actually a simple yet slick idea. *Any common divisor of two numbers* p *and* q *with* p > q *also divides* p – q. *Similarly, any common divisor between* q *and* p – q *divides* p *as well.*

    I do agree that the wording of this is pretty complex when just confronted. So, let's think about this little bit. Let p = q + xy for some real x and y. Then:

    We know that if certain positive integer divides p, it needs to divide right hand side, which includes q and xy (or p-q). So, we just showed the first part. Now, if we shift the equation to get p – xy = q, note that if certain number divides q, it must do the same for the left hand side. We know that it works for xy so same must occur for p. Thus, the Euclid's algorithm works. Probably, some of the readers are now shaking their heads in confusion. I suggest that you work this

out on your own to see how this works. Then, let's see the application example.
gcd(36,150)
= gcd(36,150-36) = gcd(36,114)
= gcd(36,78)
= gcd(36,42)
= gcd(36,6)
= gcd(30,6)….= gcd(6,6) = 6.
See how nice that was? This is the power of Euclid's algorithm. So, consider the following two terms: $36+n^2$ and $36+(n+1)^2$.

$$\gcd(36+n^2, 36+n^2+2n+1)$$
$$= \gcd(36+n^2, 2n+1)$$

This is where most students who know Euclid's algorithm stuck. Now, here comes true beauty of number theory: parity and perfect square. No matter what n is, $2n + 1$ is always odd. So, multiplying 4 to left number does not change the divisor. Why 4? We multiply by 4 because it's a key to perfect squares and in number theory, perfect squares are excellent!

$$= \gcd(4(36+n^2), 2n+1)$$
$$4(36+n^2) = 144+4n^2 = (2n+1)^2 + (143-4n)$$
$$= \gcd((2n+1)^2 + (143-4n), 2n+1)$$
$$= \gcd(143-4n, 2n+1)$$

Make sure you see how the last line in the equation came. And here, because one coefficient of n is positive and other one is negative, we don't want to directly start subtracting because that won't get us anywhere. So, let's use the similar method as above.

$$143-4n = -2(2n+1)+145$$
$$= \gcd(-2(2n+1)+145, 2n+1)$$
$$= \gcd(145, 2n+1)$$

Thus, 145 is our answer and occurs when $n = 72$. Indeed, a_{72} = 5220 = 36*145 and a_{73} = 5365 = 37*145.

15. I really like this problem because I was off by very small number on the original. The largest such number is 315, and the second largest (which I thought) was 312. I'll come back to why I got it wrong but let's see how you would solve this problem.

    Problems involving different bases are usually not *that* difficult but they are just difficult to approach. Even in this problem, the first question is, should I consider the numbers in base 10 and convert them to base 7 or work on base 7 and convert them to base 10? Or should I use algebraic methods? The latter doesn't work too well because we're finding the largest. As far as which bases, we want to do base 7 because it's smaller, and note that $1000_7 = 343_{10}$. Why is this important? Well, it tells us that numbers less than 1000 (which would be as big as 666_7) are the boundary points since numbers bigger than this will more likely be three times rather than twice.

It's okay if you do not realize it right away. I did not, and perhaps that's why I got it wrong because if you go down from 666_7, you can easily see that the largest occur at 630_7 which gives 315_{10}. Now, then what is the other method? Well, based on the problem's example, we can try to go backward and forward. The interesting thing is that we only get new doubles (usually) if the number in base 7 increases significantly. For instance, from 102_7, the next such case (if I'm not mistaken) is at 204_7 with 102_{10}. We can track this by seeing that 200_7 gives 98_{10} whereas numbers before that had significant differences.

This problem is rather difficult to just say "do that" and "this." You have to start thinking about what you should do. Do not just start writing random stuffs for problems involving different bases. Think about what you are doing and how you should do to get something that would be of an interest – this could mean lots of caseworks or small caseworks or little bit of both.

16. This is a very famous AIME problem but I had to look at the solution to get the answer. Sadly, I'm not very good at visualizing, which comes really handy here. Visualizing is one of the extra skills that you want to have for number theory problems. In many difficult number theory problems, the answer usually requires indirect epiphanies by small case works. Unlike USAMO, however, you don't need to prove this "assumption" and you are usually safe to go with your assumption. Of course, there are exceptions so this is not to be taken literally for all cases.

    Anyway, let's consider alternating sum for $n = 4$.

    We have:

    {1}, {2}, {3}, {4} for single-element sets

    {2-1}, {3-1}, {3-2}, {4-1}, {4-2}, {4-3} for two-elements sets

    {3-2+1}, {4-3+2}, {4-3+1}, {4-2+1} for three-elements sets

    {4-3+2-1} for four-elements sets

    If we focus on 3, we see that all 3's get cancelled. But how? Is this just a coincidence? This is where you want to visualize and set out "unwritten" proof and assume for other numbers (since there is nothing special about 3 other than I said 3 ☺).

If you think about it, there is +3 on the single set. For the two-elements sets, you know that there is one -3 with 4 and two +3 with 1 and 2, respectively. For the three-elements sets, you can draw on your head that there is two -3 with 4 & 1 and 4 & 2 and one +3 with 1 & 2. Lastly, there is one -3 with other three elements. So, we have total of 4 +3's and 4 -3's so they cancel.

If you do not get this, contemplate on this fact for sometime. Soon, you'll realize that this works for all numbers EXCLUDING *n*. For *n*, we can calculate the numbers of *n* by hands. But… we can also use combinatorics.

Note that for single set, this is just 1 or $\binom{n-1}{0}$. For two-elements sets, there are *n-1* or $\binom{n-1}{1}$. This continues.. To see this, make sure you work this out!

Thus, the total number of appearances of *n* is same as calculating:

$$\binom{n-1}{0}+\binom{n-1}{1}+\cdots\binom{n-1}{n-2}+\binom{n-1}{n-1}$$

This is equivalent to 2^{n-1}. If you do not believe this, use mathematical induction to prove it. Deriving is bit more difficult but it's a very famous identity that even appeared at middle school MathCounts. So, the sum is $n*2^{n-1}$. In this case, it's $6*2^5$ or 6*32 = 192. This is a very nice problem that can be solved much more quickly by visualizing and using combinatorial skills, which do come handy in many difficult number theory problems.

V. Ending Remark

How can you become a good tennis player?
You should go (not try to go) to the court and start hitting the tennis ball everyday.
You should practice your serves and develop various types of serves like slice.
You should also go to the wall and hit back and forth to work on your strokes.
You should also find a partner to play a match with or just work on your strokes.
You need to keep yourself in good condition by constantly running, avoiding unhealthy habits, and remind yourself that this is what you want to do.
You have to find a diversion other than tennis. There should be something that you can do to relax and not think about tennis for couple hours that when you go back to play, your mind is fresh.
Most of all, enjoy the every minute. It's not worth if you do not like the play.

Why did I make that?
I wanted to make a point that improving your problem solving skills in number theory or any area is like constant practice for athletics. Especially with number theory, you need to develop a high level of patience because the problems involving number theory are not obvious. There is no easy way… You just have to continue practicing and *always* have an escape route so you do not lose sanity.

Sincerely,

January 25, 2009
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