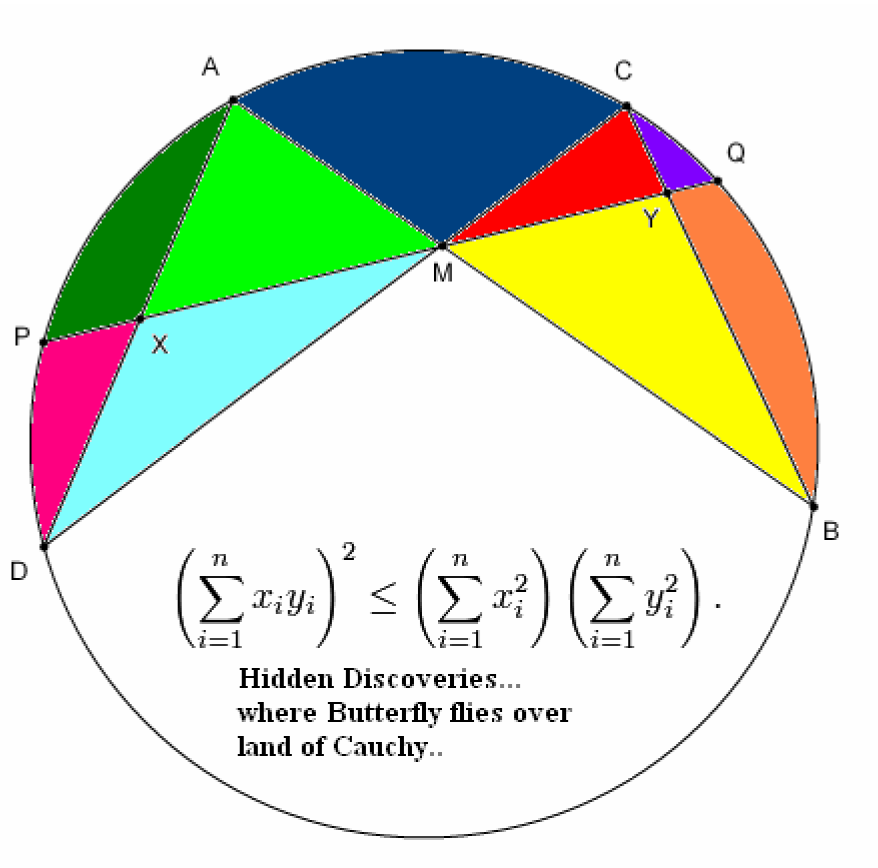


HIDDEN DISCOVERIES OPEN MATHEMATICS CHALLENGE

Rules

1. In contrast to previous round where I specified the rule, I am not going to reiterate here.
2. The rules applying to official AIME (American Invitational Math Examation) will be enforced.
3. **All times refer to Eastern Time Zone**. The problems will be sent via email on Sunday 25th of January 2009. The answers are due by 9:00 A.M. on Tuesday 27th of January 2009.
4. All emails must be to hdmatholympiad@gmail.com. You may only send answers ONCE. There is no way to change it once sent.
5. All contestants will receive solutions to the problems.

1. Let K be a positive integer such that:

$$K=\sum_{n=1}^{4}n^5-\sum_{n=1}^{4}n^4-\sum_{n=1}^{4}n^3-\sum_{n=1}^{4}n^2-\sum_{n=1}^{4}n$$

Find the remainder when K is divided by 5.

2. Consider the polynomial $P(x)$:

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_2x^2+a_1x+a_0$$

and a positive integer z:

$$z=\sqrt{3+2\sqrt{2}}-\sqrt[2010]{(1-i)^{2008}+(1+i)^{2008}}$$

$$i=\sqrt{-1}$$

If $P(z)$ and $P(z-1)$ are both odd, there are Q number of integer roots. Find $(Q+3)(Q+4)^2$.

3. Triangle ABC with $AB=13, BC=14,$ and $AC=15$, is constructed with point P inside it. P is r distance from AC, s distance from BC, and t distance from AB. If $s^2+r^2+t^2=\frac{84}{295}$ and $s=\frac{x\sqrt{y}}{z}$ where x and z are relatively prime integers and y is square-free, find $x+y+z$.

4. Let $f(x)=\frac{4^x}{4^x+2}$ and $S_n=\sum_{i=1}^{n-1}f(\frac{i}{n})$ for $n\geq 3$

For $333\leq n\leq 777$, let T be the average of all values of n such that S_n is an integer. Find T.

5. For positive integers a,b,c, find the number of all ordered triples (a,b,c) satisfying the equation:

$$ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)=12$$

6. Sam, a very patriotic American, plans to make a poster with regular octagon-shaped figure. Label this figure as $A_1A_2\ldots A_8$ with O as its center. Triangular regions $OA_mA_{m+1},\ 1\leq m\leq 8$ (obviously, $A_9=A_1$) are to be colored by three colors of American flag --- blue, red, and white. Also, to make the figure interesting, Sam wants the adjacent regions to be colored in different colors. In how many ways can Sam accomplish this?

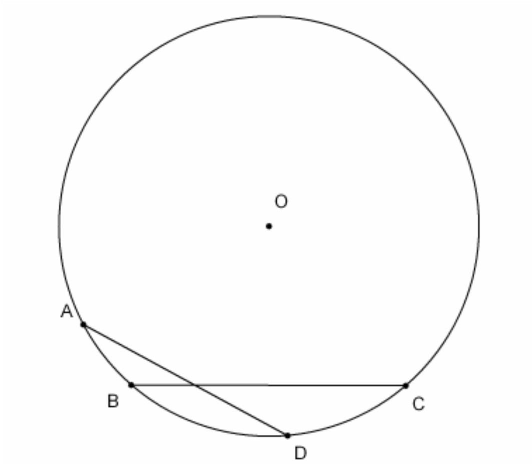
7. Polynomial $P_n(x)$ is defined like this:

$$P_n(x) = \binom{n}{2} + \binom{n}{5}x + \binom{n}{8}x^2 + \cdots + \binom{n}{3k+2}x^k,$$

where $n \geq 2$ is a natural number and $k = \left\lfloor \frac{n-2}{3} \right\rfloor$. Compute the remainder when T is divided by 1000:

$$T = 3P_{10}(2) - 3P_9(2) + 3P_8(2)$$

8. Below diagram shows circle O with two intersecting chords in it. It is given that the radius of the circle is 20, that $BC = 24$, and that BC bisects AD. Also, suppose that AD is the only chord starting at A, which is bisected by BC. It follows that cosine of the minor arc AB is a rational number $\frac{m}{n^2}$ where m and n are relatively prime integers. Find $m+n$.

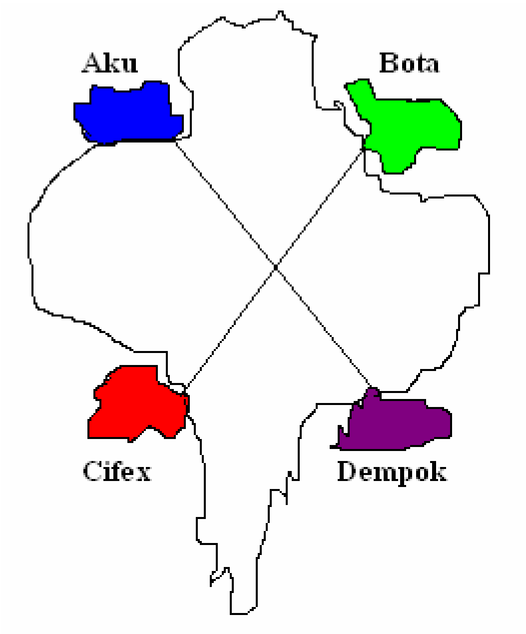


9. For integers n in the interval $0 \leq n \leq 44$, let

$$P_n = \frac{\prod_{k=0}^{44} \cos^2(2k+1) \cdot \sin^2(2n+1)}{\cos^2(2n+1)}$$

If $\frac{\sum_{i=0}^{44} P_i}{\prod_{j=0}^{44} \cos^2(2j+1)} = m$, find $\lfloor \frac{m}{10} \rfloor$ (In here, $\lfloor x \rfloor$ means the floor function and the numbers in the problem are in degree measures).

10. In the map below, Eric is stranded on mystical island called Aku. Each island --- Aku, Bota, Cifex, and Dempok --- is connected by bridges to other three islands, and each bridge has length of 1 km. From Aku, he chooses one of the three bridges from Aku, and walks along that bridge to the island at its opposite end. Let's say that by the end of the day, Eric walked 8 km solely on bridges. Let $\frac{a}{b^c}$ be the probability that Eric is back at the island Aku at the end of the day (i.e. after 8 miles of walk). If a, b, c are positive integers with $(a, b) = 1$ and b as small as possible, find $a + b + c$.



(Please note that the length of bridges is NOT drawn in scale, and there are 6 bridges in the drawing above)

11. Let v and w be randomly chosen roots of the equation $z^{2009} - 1 = 0$. Let $\frac{m}{n}$ be the probability that $\sqrt{2+\sqrt{2}} \le |v + w|$, where m and n are relatively prime positive integers. Find $m + n$.

12. In triangle $ABC, AB = 20, BC = 21,$ and $AC = 29$. A circle with center O is circumscribed about triangle ABC. Point D is chosen such that the distance between D and O is the positive root of the cubic equation

$$x^3 + x^2 - \frac{145}{4}x - \frac{145}{4} = 0$$

Let $\frac{k}{q}\pi$ be the positive difference in the area of circles with center D and center O. Find $k + q$.

13. Amber tossed a fair coin 20 times. The probability of her to have total of 6 tails and no more than 5 heads in a row can be expressed as $\frac{m}{n^2}$ where m and n are relatively prime positive integers. Find $\lfloor \frac{m-n}{10} \rfloor$ ($\lfloor x \rfloor$ represents the floor function).

14. In triangle ABC, $AB = 7, BC = 9,$, and $AC = 8$. Point K is chosen such that:

$$\angle BCK = \angle ABK = \angle CAK$$

If $\tan \angle BCK = \frac{r\sqrt{s}}{t}$, find $r + s + t$.

15. Sinister Math Wizard created two sinister (no surprise here!) functions to harass students preparing for AIME. These two functions are:

$$S_1(x) = 4\cos^3 x + 3\cos^2 x - 3\cos x + 1$$

$$S_2(x) = \cot x \cot 2x \cot 3x - \cot x - \cot 2x - \cot 3x$$

where *x* indicates measure of angle in degree. If $S_1(x) = \frac{3\sqrt{3} + 2\sqrt{2} + 10}{4}$, then find $4S_2(x)$.