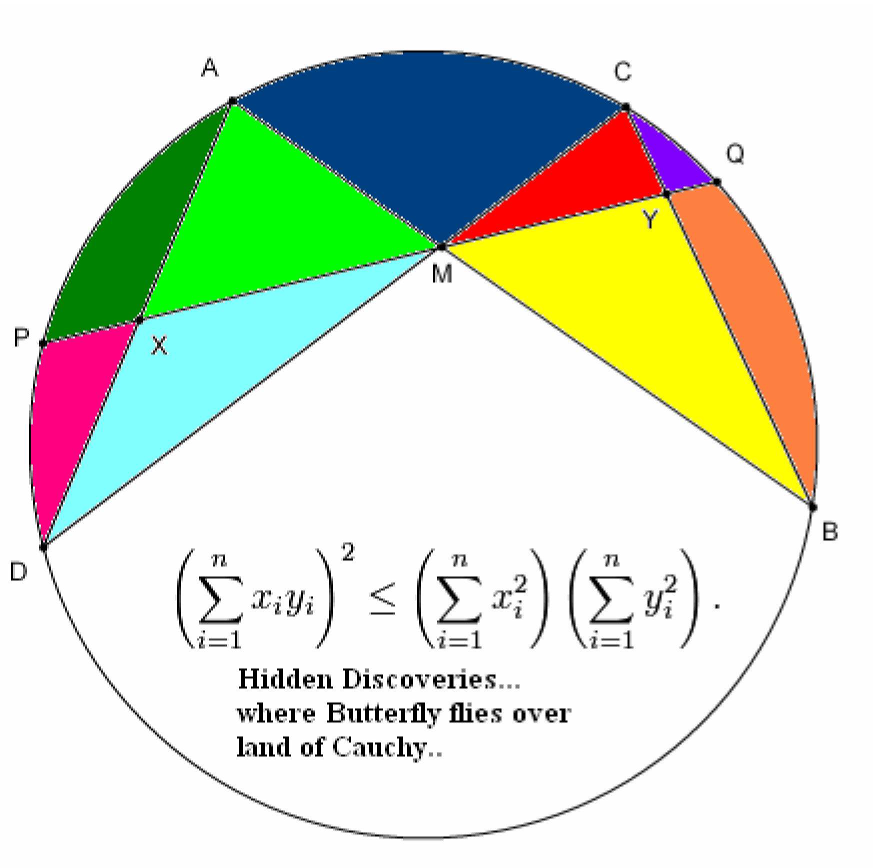


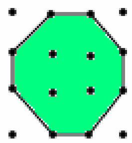
Hidden Discoveries Preliminary Round Problems

2008-2009 School Year

Rules to Follow

1. There are 25 questions in this exam with five multiple choices (A – E). Only one choice is correct.
2. The contestant is awarded 6 points for each correct answer, 1.5 for unanswered question, and 0 for incorrectly answered question.
3. The contestant is prohibited from using calculator in this exam.
4. The contestant is permitted to use pencil, eraser, ruler, scratch paper (if provided by proctor), compass, and other utensils allowed in official AMC by MAA.
5. Essentially all the rules applying to AMC by MAA will be present in this exam.
6. The contestant has one week period to finish the exam but once the exam began, the contestant must finish the exam (no returning later).
7. There are almost 100 students taking this exam internationally so please be honest and take exam in the amount of time equivalent to the actual AMC.
8. Once done, all answers must be submitted via email to hdmatholympiad@gmail.com in the one week period given (Dec. 15 to Dec. 21). Any answer received later will not be counted.
9. Those that score at least 100 or in top 30% will automatically be sent to the second round, equivalent to AIME. Note that there is no third round equivalent to USAMO. It is been cancelled.
10. No outside source such as books, internet, dictionary, or another person can be incorporated into this exam.
11. Good luck!
12. **COACHES ONLY:** Despite the email, you MUST submit the name, grade, and ALL answers of each student to the email. This is necessary for statistics purpose. You may grade the exam (since you will be provided with the answer keys) but you must still send the all 25 letter answers to the email. Thank you.

1. If each point is 1 unit away from its neighbor point vertically and horizontally, find the area of the shaded figure.



A) 7 B) $\frac{15}{2}$ C) 8 D) $\frac{81}{10}$ E) $\frac{17}{2}$

2. In golf, scoring a <u>par</u> means that you put the ball in designated number of shots. For example, a hole with par 4 means that you must finish the hole in 4 shots to be even. Making the hole in 5 shots will then mean 1 shot over, and making the hole in 3 shots mean 1 shot under. In Summer Golf Course, Grant managed following shots:

Hole Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Par	4	4	5	3	4	5	4	3	4	4	4	3	4	5	4	4	3	5
Score	3	6	2	3	7	3	3	3	3	3	5	3	4	4	5	6	6	4

By how many shots was (were) he away from par of entire course?

A) 1 B) 2 C) 3 D) 4 E) 5

3. The first and second term in the sequence are 1 and 3, respectively. For $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. What is 10^{th} term in this sequence?

A) 76 B) 77 C) 78 D) 122 E) 123

4. How many <u>distinct</u> real solutions are there to the equation

$$(x^2 - 2x + 1)(x^2 - x + 2)(x^2 + 2x - 1)(x^2 + x - 2) = 0$$

A) 0 B) 2 C) 4 D) 6 E) 8

5. A bus leaves New York City for Philadelphia every 20 minutes, starting at 6:00 A.M. A bus also leaves Philadelphia for New York City on the same schedule. All buses travel at a constant speed along the same route, and the trip either way takes 2 hours. Including the buses it meets at each terminal, how many buses will the one leaving New York City at noon encounter on its trip to Philadelphia?

A) 6 B) 7 C) 13 D) 14 E) 15

6. Ross, Sam, and Thad throw darts simultaneously at a tic-tac-toe board, and each hit a different square. What is the probability that the three hits do not mean a win at tic-tac-toe (that is, not a straight line)?

A) $\frac{4}{9}$ B) $\frac{9}{21}$ C) $\frac{6}{7}$ D) $\frac{19}{21}$ E) $\frac{20}{21}$

7. An equilateral triangle ABC of side length 1 is inscribed in an circle. Point P is chosen in the minor arc BC such that PA is perpendicular to BC. What is the value of PA + PB + PC?

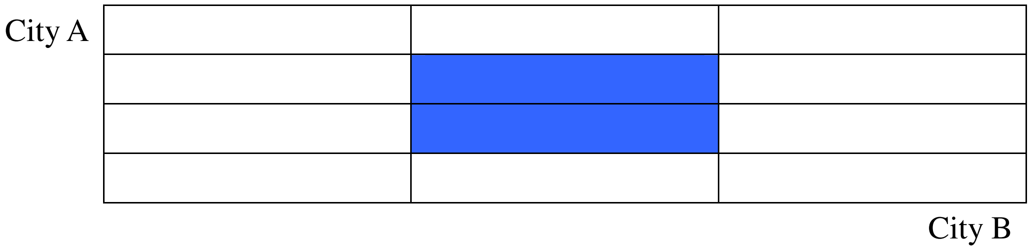
A) $\sqrt{3}$ B) $\frac{4\sqrt{3}}{3}$ C) 3 D) $\frac{256}{81}$ E) $3\sqrt{3}-2$

8. Find $x^2+2xy+y^2$ if

$$x\sqrt{2}+y\sqrt{3}=1$$
$$x\sqrt{3}+y\sqrt{2}=2$$

A) $\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$ B) $\frac{9}{5+2\sqrt{6}}$ C) 1 D) $\frac{3+2\sqrt{2}}{4}$ E) $\frac{3}{2}$

9. In how many ways (shortest as possible) can a person travel from City A (top left vertex) to City B (rightmost bottom vertex) if a person cannot go across (around on borders is fine) the shaded area?



A) 24 B) 25 C) 26 D) 27 E) more than 27

10. In a parabola $y = x^2$, vertex and two of its points are chosen to form a right triangle ABC with C as a right angle and vertex. If A is on Quadrant II and the area of triangle is 1, what is the sum of B's x-coordinates and y-coordinates?

A) 1 B) $\sqrt{2}$ C) $\sqrt{3}$ D) 2 E) $\sqrt{5}$

11. Farmer John has piles of apples comprising of one pile of 5 apples, one pile of 3, and one pile of 1. Such a collection can be denoted as (5,3,1) with abbreviation A_1. A new collection is obtained by harvesting A_1 where harvesting is defined to remove one apple from each pile to form a new pile. Hence, A_2 = (4,3,2), and if A_2 is harvested, A_3 = (3,3,2,1). Find A_{2008}.

A) (5,3,1) B) (4,3,2) C) (3,3,2,1) D) (4,2,2,1) E) (4,3,1,1)

12. Find the positive value of *x* that satisfy the equation:

$$\frac{1}{x-11} - \frac{1}{x-7} = \frac{1}{x+25}$$

A) 23 B) 24 C) 25 D) 26 E) 27

13. Let *a,b,c* be real numbers defined as below.

$$a = \cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{23\pi}{12}\right)$$

$$b = \cos\left(\frac{\pi}{12}\right)\sin\left(\frac{23\pi}{12}\right)$$

$$c = a^2 - 2b$$

Which interval contains *c*?

A) [-0.5,0) B) [0,0.5) C) [0.5,1) D) [1,1.5) E) [1.5,2)

14. The six edges of tetrahedron ABCD are 8,14,19,28,37, and 42. If the length of edge AC is 42, find the length of BD.

A) 8 B) 14 C) 19 D) 28 E) 37

15. For all real values of *x*, *f(x)* satisfies:

(i) $f(x) + f(2-x) = 7$

(ii) $f(2+x) = 2 + f(x)$

Then $f(x) + f(-x)$ must be

A) 7 B) 6 C) 5 D) 4 E) 3

16. Let $(a_1, a_2, a_3, \cdots a_n)$ be a permutation of $(1, 2, 3, \cdots n)$ for which:
$a_1 > a_2 > a_3 > \cdots > a_{\frac{n}{2}}$ and $a_{\frac{n}{2}} < a_{\frac{n}{2}+1} < a_{\frac{n}{2}+2} < \cdots < a_n$

for n as an even positive integer;
$a_1 > a_2 > a_3 > \cdots a_{\frac{n+1}{2}}$ and $a_{\frac{n+1}{2}} < a_{\frac{n+1}{2}+1} < a_{\frac{n+1}{2}+2} < \cdots a_n$

for n as an odd positive integer.

Let the total number of permutations of n be $P(n)$. For how many values of n, is $200 < P(n) < 500$?

A) 1 B) 2 C) 3 D) 4 E) 5

17. Define *alpha function* as:
$\alpha_n = (\log_{a_1} b_1)(\log_{a_2} b_2)(\log_{a_3} b_3)\cdots(\log_{a_n} b_n)$
where $a_1 = 2, a_2 = 3, a_3 = 4, \cdots a_n = n+1$ and

$b_1 = 3^2, b_2 = 4^2, b_3 = 5^2, \cdots b_n = (a_n + 1)^2$. What is α_{2007}?

A) $2^{2008} \cdot \log_2 2007$ B) $2^{2007} \cdot \log_{2007} 2$ C) $2^{2008} \cdot \log_2 2008$

D) $2^{2008} \cdot \log_{10}(2008 + 2^{2006})$ E) $2^{2007} \cdot \log_2 2009$

18. On $\triangle ABC$, points D and E are chosen on AB and BC, respectively, such that $BD : AD = 1 : x$ for some positive integer x, and $BE : CE = 2 : 5$. CD and AE intersect at F. If $\frac{DF}{CF} = \frac{3}{10}$, what is x?

A) 1 B) 2 C) 3 D) 4 E) 5

19. Let $N = \sum_{i=0}^{9} (-1)^i \cdot 2^{2^i}$. Find the sum of the digits of N when N is written in binary.

(A) 191 (B) 273 (C) 288 (D) 329 (E) 341

20. For integers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$:

$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = 7$
$a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3 = 8$
$a_1b_3 - a_3b_1 + a_4b_2 - a_2b_4 = 9$
$a_1b_4 - a_4b_1 + a_2b_3 - a_3b_2 = 10$

What is $(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2)$?
(A) 100 (B) 105 (C) 120 (D) 225 (E)294

21. Aaron has four ropes. He selects two of the eight loose ends at random (they can be from the same rope), and ties them together, leaving six loose ends. He again selects two loose ends at random, and so on, until there are no loose ends. Find the expected value of the number of loops that Aaron ends up with.

(A) $\frac{177}{106}$ (B) $\frac{176}{105}$ (C) $\frac{175}{104}$ (D) $\frac{174}{103}$ (E) $\frac{173}{102}$

22. In *xy*-plane, a circle with center O with equation $x^2 - 26x + y^2 - 168y = 51^2 - 13^2 - 84^2$ is drawn. Let A be the origin. Then, there exists a point B on this circle such that $\angle ABO = 90°$. Find the slope of $\overline{AB}$.

(A) $\frac{-15}{8}$ (B) $\frac{-3}{2}$ (C) $\frac{-133}{111}$ (D) $\frac{2}{5}$ (E) $\frac{133}{111}$

23. A sequence $(a_1, b_1), (a_2, b_2), (a_3, b_3), \cdots$ of points in the coordinate plane satisfies $(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n)$ for $n = 1, 2, 3, \cdots$
If $a_1 = b_1 = 1$ and $(a_{52}, b_{52}) = (m, n)$, then what is $m + n$?
(A) -4 (B) -3 (C) -2 (D) -1 (E) 0

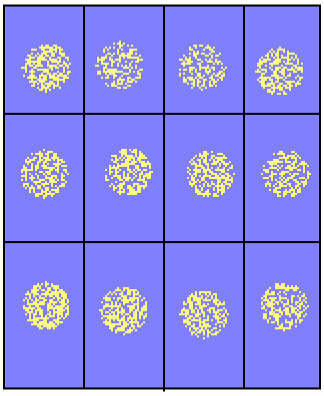
24. Let $x_1, x_2, \cdots x_n$ be a sequence of integers such that
(i) $-1 \le x_i \le 2$, for $i = 1, 2, 3, \cdots n$;
(ii) $x_1 + x_2 + \cdots + x_n = 20$;
(iii) $x_1^2 + x_2^2 + \cdots + x_n^2 = 80$.
Find the maximum possible value of

$$\left(x_1^4 - x_n^3\right) + \left(x_2^4 - x_{n-1}^3\right) + \cdots \left(x_{n-1}^4 - x_2^3\right) + \left(x_n^4 - x_1^3\right)$$

(A) 132 (B) 144 (C) 156 (D) 168 (E) 180

25. The Joker is haunting the Gotham City with his twelve deadly cards, which are 4 A's, 4 Kings, and 4 Queens. Each type of card comes in four suits – heart, spade, club, and

diamond. The cards are placed in three rows as below:



As shown above, the cards will have their faces down and only the Joker will know how they are placed. But, he did tell the citizens of the Gotham City this:

- In the first row, none of the four cards is A.
- In the second row, none of the four cards is King.
- In the third row, none of the four cards is Queen.

The Joker also said that of all the combinations as suggested above, there is actually one that is harmless!

If the safe format is below (no shape is drawn, however):

K	Q	K	Q
A	Q	Q	A
A	K	A	K

and the probability of choosing it of all the combinations (each has equal choice) is $\frac{1}{P}$, find P.

(A) 346 (B) 347 (C) 348 (D) 349 (E) 350