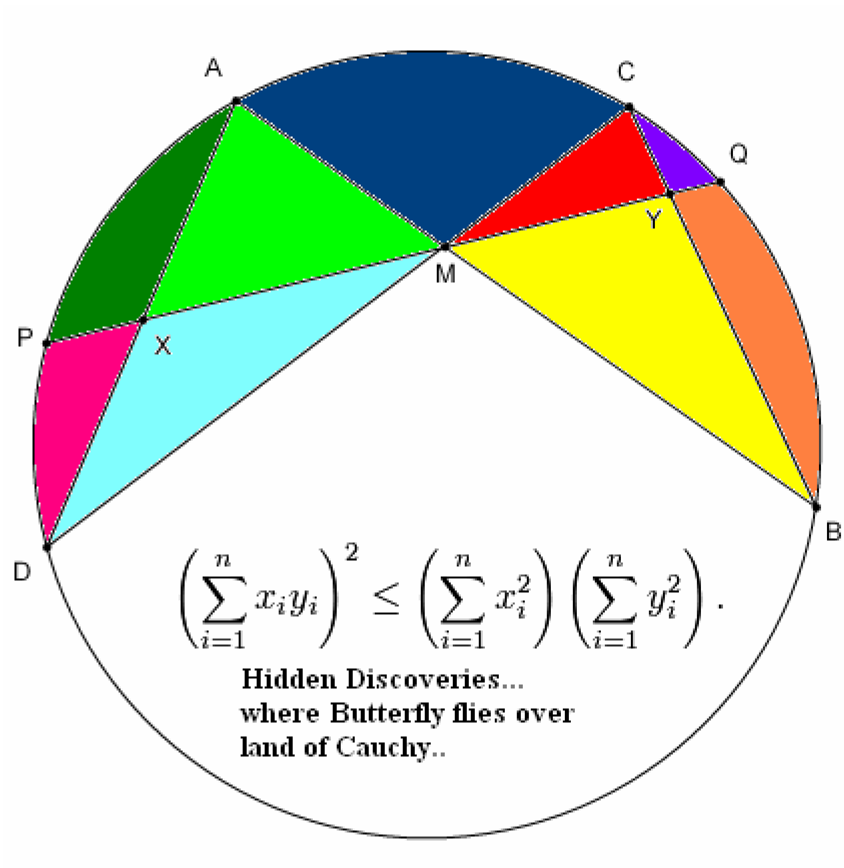


Hidden Discoveries Preliminary Round Solution

This PDF document will list at least ONE solution to each of 25 problems. By no means, do these solutions denote the best solution but rather, the one that the problem maker used. The whole point of solution is both to show how the problem is done and to explain the methods of solving them. It is highly recommended that the contestant learns from these solutions.

1. (A) The total area of the grid is $3^2 = 9$ square units since it's a square. To find the shaded figure, subtract the areas of four right triangles on the corner. Each has area of 1*1*1/2 = 1/2. So:
   A = 9 – 4(1/2) = 9 – 2 = 7

   <u>OR</u>

   By Pick's Theorem, A = I + B/2 – 1. There are 4 inner points and 8 boundary points.
   A = I + B/2 – 1 = 4 + 8/2 – 1 = 4 + 4 – 1 = 7

   <u>OR</u>

   The shaded figure can be divided into two congruent trapezoids with base of 1 and 3 and height of 1, and a rectangle with sides 1 and 3. So:
   A = 1*3 + 2[(1+3)*1*1/2] = 3+4 = 7

2. (A) Total par of the course is 72. By adding Grant's shots, we got 73. So, he was 1 shot from par of the course.

3. (E) Simply start adding them. The sequence (after first term) looks like
   4,7,11,18,29,47,76,123,…
   So, 123 is the answer.

   <u>OR</u>

   This sequence is Lucas number sequence. Each number in Lucas sequence can be calculated by the formula:

   $$L_n = \left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

   $L_{10} = \left[\left(\frac{1+\sqrt{5}}{2}\right)^{10} + \left(\frac{1-\sqrt{5}}{2}\right)^{10}\right] = 123$. Actual calculation can be done by using binomial theorem (this is readers to do).

4. (C) The distinct real roots are -2, 1, $-1+\sqrt{2}$, and $-1-\sqrt{2}$.

$(x^2-2x+1)=(x-1)^2$

$(x^2-x+2)=$ nonreal roots: $\left(\frac{1+i\sqrt{7}}{2}\right)$

$(x^2+2x-1)=$ two real roots: $-1\pm\sqrt{2}$

$(x^2+x-2)=(x-1)(x+2)$ BUT x-1 is already counted so only -2.

5. (C) First of all, it is necessary to note that there are six intervals: 12:00 to 12:20, 12:20 to 12:40, etc, to 1:40 to 2:00. Now, when the bus leaves the terminal, other bus from Philadelphia enters the terminal. This is the first bus. At 12:10, the bus meets the other bus from Philadelphia, and this bus started 20 minutes after the first bus. At 12:20, the bus meets the third bus, which started 20 minutes after the second bus (and note that by then, the second bus entered the terminal on New York City). Using this idea, the bus meets another bus every 10 minutes so it meets total of 13 buses.
6. (D) P(event) = 1 – P(not event)
   Let's say "event" is "not win" and "not event" is "win." There are 9C3 = 84 ways to choose 3 square out of 9 squares. To form a win, it must be 1 of 3 horizontal lines, 3 vertical lines, or 2 diagonal lines. So:
   P(not win) = 1 – P(win) = 1 – 8/84 = 76/84 = 19/21.
7. (B) Let PA and BC meet at D. Then right triangle ABD is similar to the right triangle APB. So, by proportion:

   $$\frac{AB}{AP}=\frac{BD}{PB}$$

   $$\frac{1}{AP}=\frac{\frac{1}{2}}{PB}$$

   Now, PB = PC by symmetry so PA + PB + PC = PA + 2 PB = PA + 2(PA/2) = 2PA.
   Using 30-60-90 triangle ratio, since AB (opposite of 60 degrees angle) is 1, then the hypotenuse must be $\frac{2\sqrt{3}}{3}$. So, the answer is $2\left(\frac{2\sqrt{3}}{3}\right)=\frac{4\sqrt{3}}{3}$.

   <u>OR</u>

   You can find the length of PB individually using Pythagorean Theorem and/or similar triangles.

OR

Applying Ptolemy's Theorem on quadrilateral ABPC:

PC*1 + PB*1 = PA*1

PA = PB + PC

Then, applying Power of a Point Theorem:

1/2*1/2 = AD*DP

AD = $\frac{\sqrt{3}}{2}$ (height of triangle ABC)

DP = $\frac{\sqrt{3}}{6}$

Thus, PA = $\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{6}=\frac{2\sqrt{3}}{3}$ and PA+PB+PC = $\frac{4\sqrt{3}}{3}$.

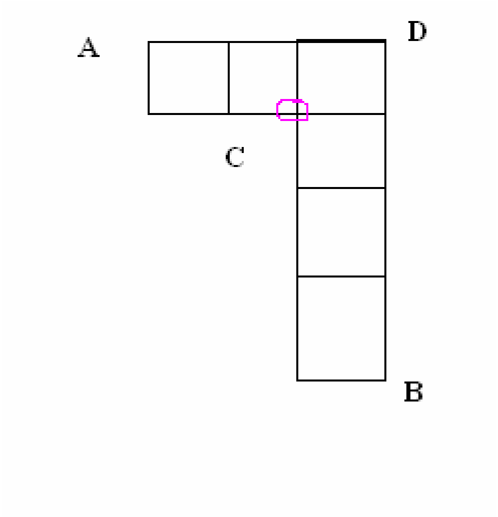
8. (B) Add them.

$$x\left(\sqrt{2}+\sqrt{3}\right)+y\left(\sqrt{2}+\sqrt{3}\right)=3$$
$$\left(\sqrt{2}+\sqrt{3}\right)[x+y]=3$$
$$x+y=\frac{3}{\sqrt{2}+\sqrt{3}}$$
$$x^2+2xy+y^2=(x+y)^2$$

So, the answer is:

$$\left(\frac{3}{\sqrt{2}+\sqrt{3}}\right)\left(\frac{3}{\sqrt{2}+\sqrt{3}}\right)=\frac{9}{5+2\sqrt{6}}$$

9. (C) Note that the number of ways to go top of a shaded area is same as the number of ways to go below the shaded area. So, consider the following diagram (the square are *supposed* to be equal in size but that is not how they appear so please allow that difference to sink in):



There is only 1 way to go from D from A to B. From A, there are 3 ways to go to C. From C, there are 4 ways to go to B. So, there are total of 2(1+3*4) = 2(1+12) = 26 ways to go from A to B. Note that we had to multiply by 2 because of the symmetry that I stated earlier in the solution.

10. (D) Let A be (a,a^2) and B be (b,b^2). Because ABC is a right triangle, AC is perpendicular to BC. Using the slope:

$$\frac{b^2}{b}\cdot\frac{a^2}{a}=-1\rightarrow ab=-1$$

Since A is on Quadrant II, a < 0 so b must be positive. Using the distance formula:

$$\frac{AC\cdot BC}{2}=1$$

$$AC\cdot BC=2$$

$$\left(\sqrt{a^2+a^4}\right)\left(\sqrt{b^2+b^4}\right)=2$$

Letting a = -1/b:

$$\left(\sqrt{\frac{1}{b^2}+\frac{1}{b^4}}\right)\left(\sqrt{b^2+b^4}\right)=2$$

$$\sqrt{2+\frac{1}{b^2}+b^2}=2$$

$$2+\frac{1}{b^2}+b^2=4\rightarrow\frac{1}{b^2}+b^2=2$$

From here, b is clearly 1 or -1. But since b must be positive, B has coordinates of (1,1) and the answer is 1+1 = 2.

<u>OR</u>

Trying out few values can help you to be lucky and get the right answer (beauty of brute forcing?)

11. (D) The cycle is:

$C_2 = (4,3,2)$

$C_3 = (3,3,2,1)$

$C_4 = (4,2,2,1)$

$C_5 = (4,3,1,1)$

$C_6 = (4,3,2)$

……

So, $C_{2008} = C_4 = (4,2,2,1)$.

12. (A) Let y = x-7 then the equation is:

$$\frac{1}{y-4} - \frac{1}{y} = \frac{1}{y+32}$$

Solving this simple fraction yields $y^2 - 8y - 128 = 0$. $y^2 - 8y - 128 = (y-16)(y+8) = 0$ → y = 16,-8. Now, x then is 23 or -1. We want positive so, x = 23.

13. (D) Recall that $\cos\theta = \cos(2\pi - \theta)$. So, $\cos\frac{\pi}{12} = \cos\frac{23\pi}{12}$. Thus:

$$c = a^2 - 2b$$
$$= \left(\cos\frac{23\pi}{12} + \sin\frac{23\pi}{12}\right)^2 - 2\cos\frac{23\pi}{12}\sin\frac{23\pi}{12}$$
$$= \cos^2\frac{23\pi}{12} + \sin^2\frac{23\pi}{12}$$
$$= 1$$

which is from $\cos^2\theta + \sin^2\theta = 1$ (Pythagorean Identity). Only interval [1,1.5) contains 1.

<u>OR</u>

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$
$$\sin\frac{23\pi}{12} = -\sin\frac{\pi}{12} = -\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$$

So, $c = a^2 - 2b \rightarrow a = \frac{\sqrt{2}}{2}$ and b = -1/4. Then:

$$= \left(\frac{\sqrt{2}}{2}\right)^2 - 2\left(-\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{2} = 1.$$

14. (B) Considering two triangles that AC is on, one of the sides of triangle ACD is 37 while the one of the sides of triangle ABC is 28. Using Triangle Inequality, triangle ABC must have sides of length 19, 28, and 42. So, BD is either 8 or 14. But if BD is 8, one face of tetrahedron will not fill the Triangle Inequality so BD must be 14.

15. (C) Since f(x) holds for all x, same must occur for –x.

f(-x) + f(2+x) = 7

f(2-x) = 2 + f(-x)

---

f(2+x) + f(2-x) = 9

Now, f(x) + f(2-x) = 7 so:

[f(x) + f(2-x)] + [f(-x) + f(2+x)] = 14

f(x) + f(-x) = 14 – [f(2+x) + f(2-x)]

= 14 – 9 = 5

The actual function is f(x) = x + 5/2.

16. (B) $a_{\frac{n}{2}}$ and $a_{\frac{n+1}{2}}$ are both 1. Now, note that because of 1-to-1 correspondence, the number of ways to choose numbers left of $a_{\frac{n}{2}}$ and $a_{\frac{n+1}{2}}$ (that is, $a_1 > a_2 > .... > a_{\frac{n}{2}}; a_{\frac{n+1}{2}}$) is same as that of right. So, from here is simple calculation from random n. I started with 9 because it seemed reasonable.

n = 9 → 8C4 = 70 No, too small

n = 10 → 9C4 = 126, No, still small

n = 11 → 10C5 = 252 YES

n = 12 → 11C5 = 462 YES

n = 13 → 12C6 = 924, NO TOO BIG

So, there are two values of n that work (11 and 12).

17. (E) A famous logarithm identity says:

$$(\log_a b)(\log_c d) = (\log_a d)(\log_c b)$$

So, by shifting each b_n to $_{bn+1}$, we get:

$$\left(\log_2 2009^2\right)\left(\log_3 3^2\right)\left(\log_4 4^2\right)\ldots\left(\log_{2008} 2008^2\right)$$
$$=\left(\log_2 2009^2\right)\cdot 2^{2006}$$
$$=2\log_2 2009\cdot 2^{2006}$$
$$=2^{2007}\log_2 2009$$

18. (C) Construct DG such that G is on AE and parallel to BC. So, $\triangle ADG$ ~ $\triangle ABE$ and $\triangle DGF$ ~ $\triangle CEF$. Now:

$$\frac{AD}{AB}=\frac{DG}{BE},\frac{DG}{CE}=\frac{DF}{CF}$$

So:

$$\frac{DF}{CF}=\frac{AD\cdot BE}{AB\cdot CE}=\frac{x\cdot 2}{(x+1)\cdot 5}=\frac{3}{10}$$
$$\frac{x}{x+1}=\frac{3}{4}$$
$$x=3$$

OR

I haven't worked this problem out with mass point but it might be possible using that as well.

19. (E) (Problem by Palmer Mebane – MellowMelon) The sum of the digits of 2^a-2^b written in binary ($a>b$) is $a-b$. We can pair up the numbers in the summation so that no carrying issues result by adding these differences. If we pair $i=8,9$, $i=6,7$, etc. then those differences have 1s in different places in binary. So:

$$\begin{aligned}\text{Answer} &= (2^9-2^8)+(2^7-2^6)+\ldots+(2^1-2^0)\\ &= 4^4+4^3+\ldots+4^0\\ &= \frac{4^5-1}{4-1}\\ &= \frac{1023}{3}=341\end{aligned}$$

20. (E) The answer is simply squaring each of 4 equations and adding them.

$$7^2+8^2+9^2+10^2=294$$

The problem is actually a direct application of Euler's Identity (read Art of Problem

Solving Volume I's ending "Special Factorization" or something). To have seen this, note the way the coefficients are placed. If you ponder this for a while, it is easy to see how they'll cancel. Remember, this is AMC, **not** USAMO. You don't need to *prove* what you believe. Just go with it and take a guess (and it does work in a lot of AMC problems)!

21. (B) This problem is USAMTS Year 17's problem about George and the ropes, but this time, the problem is with only four ropes (to make calculations nicer). The problem can be considered as this way:

$$E(n) = E(\text{same end}) + E(\text{different end})$$
$$= \frac{1}{2n-1}\cdot(1+E_{n-1}) + \frac{2n-2}{2n-1}\cdot E_{n-1}$$
$$= E_{n-1} + \frac{1}{2n-1}$$

So, solving recursively, $E_1 = 1, E_2 = \frac{4}{3}, E_3 = \frac{23}{15}$, and $E_4 = \frac{176}{105}$.

*NOTE: It is really difficult for me to find a hard counting problem so I took the problem from USAMTS. I know, this is not really an original problem so I apologize for this but again, counting/combinatorial problems are just SO hard to make. Furthermore, this problem is not "quite" AMC-12 problem. You will not find problems like this, involving the expected value, these days but then there was one expected problem back in late 80's (if you want to look for it, it's about boys and girls being arranged and #30 on AHSME I believe).

22. (A) Right angle implies that AB is tangent to the circle, which has the correct form $(x-13)^2+(y-84)^2=51^2$. Let C be point on x-axis below the center, which will mean that C has coordinate of (13,0). So, $\triangle AOC$ is a right triangle with AC = 13, OC = 84, and $AO = \sqrt{13^2+84^2} = 85$. By default, OB = 51 and so $AB = \sqrt{85^2-51^2} = 68$. Note that AOB is just 3-4-5 right triangle by constant of 17.

Now, recall that slope is same as tangent (if you don't know this, draw a line from origin to a point and see this).

$$m \text{ of } \overline{AB} = \tan(\angle BAO + \angle OAC)$$
$$= \frac{\tan\angle ABO + \tan\angle OAC}{1-\tan\angle BAO\cdot\tan\angle OAC}$$
$$= \frac{\frac{3}{4}+\frac{84}{13}}{1-\frac{3}{4}\cdot\frac{84}{13}} = \frac{\frac{375}{52}}{\frac{-50}{13}} = \frac{-15}{8}$$

23. (E) Let $z_n = a_n + ib_n$.

$z_n(\sqrt{3}+i) = (\sqrt{3}+i)(a_n + ib_n) = z_{n+1}$

Thus, $z_n = z_1(\sqrt{3}+i)^{n-1}$.

$z_{52} = z_1(\sqrt{3}+i)^{51}$

$= z_1 \cdot 2^{51} \cdot \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{51}$

$= z_1 \cdot 2^{51} \cdot \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{51}$

Using Demoivre's Theorem:

$= z_1 \cdot 2^{51} \cdot \left(\cos\frac{51\pi}{6} + i\sin\frac{51\pi}{6}\right)$

$= z_1 \cdot 2^{51} \cdot i$

Now:

$m + in = (a_1 + ib_1) \cdot 2^{51} \cdot i$

$= i2^{51} a_1 - 2^{51} b_1$

$= i2^{51} - 2^{51}$

$m = -2^{51}, n = 2^{51}$ so $m + n = 0$.

24. (C) Let a, b, c be number of -1, 1, and 2. 0 is out of concern. So:

$-a + b + 2c = 20$

$a + b + 4c = 80$

From these, we got

$b = 50 - 3c$ So $0 \le c \le 16$

$a = 30 - c$

Now, the given equation of problem can be arranged like this:

$(x_1^4 + x_2^4 + \cdots + x_n^4) - (x_1^3 + x_2^3 + \cdots + x_n^3)$

$= \sum_{i=1}^{n} x_i^4 - \sum_{i=1}^{n} x_i^3$

From here is just algebra with maximizing $\sum_{i=1}^{n} x_i^4$ and minimizing $\sum_{i=1}^{n} x_i^3$.

But note that the same values of a,b,c must exist so:

$$\sum_{i=1}^{n} x_i^4 - \sum_{i=1}^{n} x_i^3 = 80 + 12c - (20 + 6c)$$
$$= 60 + 6c$$
To get the maximum, let $c = 16$
$$= 60 + 6c = 60 + 6(16) = 156$$

25. (A) This is a fun problem.
For the first row, let's say there are *y* Kings, then *4-y* Queens.
For the second row, there are *y* Queens and *4-y* A's.
For the third row, there are *y* A's and *4-y* Kings.
This satisfies the problem's requirement and mathematically, this is calculated like this:
$$\sum_{k=0}^{4} \binom{4}{k}^3 = \binom{4}{0}^3 + \binom{4}{1}^3 + \binom{4}{2}^3 + \binom{4}{3}^3 + \binom{4}{4}^3$$
$$= 1 + 4^3 + 6^3 + 4^3 + 1 = 346$$
.

Note to the last three problems: Number 23 is based on 2008 AMC-12 A #25. Number 24 is based on 1999 AHSME #28. Number 25 is based on 2003 AMC-12 A #20. I realize that the number of problems actually go down in this exam but to be honest, the last three problems are about same level. Moreover, Number 23 is almost exactly like the actual problem while Number 25, although the math calculation is same (with different number), has entirely different problem to it.

Furthermore, Number 25's mentioning about the Joker and the Batman is not affiliated with the actual thing or the comic. These two characters are simply used to make the problem more interesting and unique.