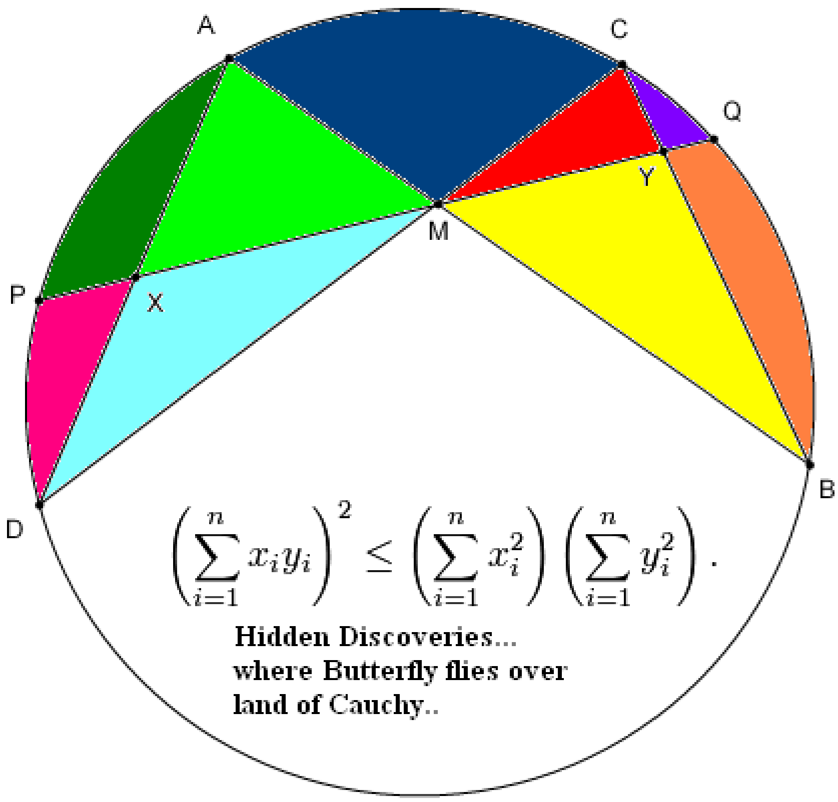
Problem Solving Strategy



First Article

I. Introduction

The motivation to write this article came from numerous and repetitive questions that came up in Art of Problem Solving forum. Every year, there is always someone who wants to know, "How can I ace AMC?" or "What books should I buy?" I, and clearly, many older members of the forum, became exhausted of seeing same questions every year. So, I decided to write this article as the first article for the competition.

Also, the problems used in this article belong to MAA and AMC. I am using for 100% free, educational purpose use, and these works should not be sold by others for profit (that'll be violating their rules).

II. Caution

As with anything in the world, keep in mind that what works for some people may not work for others. Also, what happens to someone in 3 days might require 3 months for others. There is no "magic" to the success…

III. Mind

The most important thing to keep in mind about problem solving is that most problems in AMC and definitely AIME and USAMO do not show solutions right away. Of course, there are some questions that are exceptions (especially if you've seen similar problems before) but most of them require some types of investigation and playing with smaller cases to understand.

Instead of merely "saying" them (*anyone* can do THAT!), I'll show the plans that you can use by giving example problems. You'll hopefully learn from the given strategy *and* from the problems that come from actual competition.

III. Investigation

Even the most experienced problem solvers need to investigate the problem before they can solve it. In other words, there is no such thing as solving without any work. Of course, with increasing experience, some say:

"By intuition, it is clear that…"

Now, that doesn't happen often in most people's level so investigation is really critical. Let's take a look at this problem.

(1987 AIME #14)

Compute:

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}$$

Solution:

This is a really famous problem to anyone familiar with AIME (mainly because AIME started on year 1983 and everyone usually sees problems in 1983 and 1984...). Anyway, I can tell you this: you'll probably not see straightforward problem like this in current AIME. There are several reasons such as security reason as some idiots would use calculator and lack of uniqueness but it does still provide a nice example for the first strategy.

Each () is in the form $a^4 + 4b^4$, where b = 9. Now, why didn't we use a^4+b^4? That is where investigation came in. By inspecting, we notice that the latter cannot be expanded. But, the former does… Factoring and playing with exponents is one of the major keys of investigation:

$a^4+4b^4 = (a^2)^2+(2b^2)^2+2*a^2*2b^2-2*a^2*2b^2$

$= (a^2+2b^2)^2-(2ab)^2$

$= (a^2-2ab+2b^2)(a^2+2ab+2b^2)$

Note that this way of factoring is relatively common. Adding the same thing and subtracting the same thing (thus, not changing the value). Now, let's take a look at that identity again:

$a^4+4b^4 = (a^2-2ab+2b^2)(a^2+2ab+2b^2)$

This is called Sophie Germain Identity. It is the key to this problem because this'll *almost* simplify everything!

Now, we're not quite done *yet*. We have to incorporate b = 3 to get:

(x(x-6)+18)(x(x+6)+18)

So:

$$\frac{[(10(10-6)+18)(10(10+6)+18)][(22(22-6)+18)(22(22+6)+18)]\ldots[(58(58-6)+18)(58(58+6)+18)]}{[(4(4-6)+18)(4(4+6)+18)][(16(16-6)+18)(16(16+6)+18)]\ldots[(52(52-6)+18)(52(52+6)+18)]}$$

$$=\frac{(10(4)+18)(10(16+18))(22(16)+18)(22(28)+18)\ldots(58(52)+18)(58(64)+18)}{(4(-2)+18)(4(10)+18)(16(10)+18)(16(22)+18)\ldots(52(46)+18)(52(58)+18)}$$

Almost all terms cancel, or *telescope*, except the first term in the denominator and the last term in the numerator:

$$\frac{58(64)+18}{4(-2)+18}=\frac{3730}{10}=373$$

Now, let's talk about the significance about this problem. Although this problem may seem "out of nowhere" (that is, who would've used this kind of identity?!) but it is actually pretty popular. It appeared in *Problem-Solving Strategies* by Arthur Engel, and he gave two example problems using this:

1. (1978 Kurschak Competition) For n>1, prove that n^4+4^n *is never a prime*.
2. (Russian Olympiad for 8th grader) Is $4^{545} + 545^4$ a *prime*?

I'll leave that to readers to have "fun" with. But regardless, you should now understand the importance of investigation, or some call it, *playing around with numbers*.

IV. Have a Strategy Part 1

Some problems have pretty straightforward strategy. With geometry problems involving triangles and area, some "strategies" include drawing altitude, parallel lines for perpendicular angles or congruent angles, or extension of existing lines to form similar triangles, and so on. With algebra problems, factoring or considering smaller cases, or anything that comes from *investigation* comes in handy. But, then, there are those that you look at say, "Okay, I know what the problem is saying but WHERE do I start?" This is for those who ask this question: have a strategy.

As always, my words are not nothing but alphabets so…

(1985 AIME #14) In a tournament each played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned ½ point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten). What was the total number of players in the tournament?

Solution:

Anyone familiar with *handshake* problem (ex. *X number of people came to the party and shake each other exactly once; then how many handshakes take place?*) should realize that this problem is very similar to it. Note that the total number of points equals the total number of games, or "handshakes," that were taken place (make sure you understand this; this is constant regardless of win/loss or ties because the SUM of the points is always 1 in all the games!).

Now, this was the "strategy" in this problem: to visualize the problem as something bit easier. Now, next strategy is to use the problem's wording: *After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points*. Why is this important? Well, this lets you to find the sum of the points in two ways: by total at once or by adding the 10 lowest scorers' and the rest of the players' points. Anytime you are faced with problems like this where something is equal to another, always think to set two parts equal. That's usually the "key."

With handshake problem, the number of handshakes is in form n(n-1)/2 where n denotes the number of people. So, using this, the total number of points (using same variable) is:
n(n-1)/2

Now, if there are n people as above, then there are n-10 people other than the lowest ones. We can find the total number of points for the "rest" and 10 lowest by considering each of the points among themselves and multiplying that by 2. With the "rest," the number of points is (n-10)(n-11)/2 so the total number of "rest" points is (n-10)(n-11) [multiplied by 2]. Similarly, the number of points for 10 lowest is 9*10/2 = 45 points so the total number of 10 lowest is 45*2 = 90 points. Now, let's use algebra!

$$\frac{n(n-1)}{2} = 90 + (n-10)(n-11)$$

$$n^2 - 41n + 400 = 0$$

$$(n-16)(n-25) = 0$$

$$n = 16,25$$

Now, here are two key points that go along with having strategy.

- Watch out for brute-force. Here, we see that the number of players is either 16 or 25 ----- clearly, too much for brute-force. For those that do not know this term, *brute-force* is simply doing everything by hand. This is OK for investigation (just to see how the problem turns out) but note that with strategy, this is no-no because it doesn't get you anywhere. Sometimes, yes, but probably not always.
- CHECK YOUR ANSWER… Obviously, checking your steps (if you have time) helps too but you should always check your answers. Why? Let's see…

If there were 16 players, then there are 6 non-lowest players and 10 lowest players. Their total number of points is then 6*5/2*2 = 30 points, and because there are 6 players, the average point is 30/6 = 5 points. Now, with the 10-lowest players, the total points are 90 so the average point is 90/10 = 9 points. But this means that the 10-lowest players are actually not the lowest ones! Contradiction.

Now, with 25 people, it works. One scenario is like this:

- 10 lowest and 15 rest
- Ties among them → lowest points among them = 9/2 = 4.5; rest points among them = (15*14/2)/15 = 7 points (note that how this worked out…)
- Ten games played by each player in the "rest," let there be 6 wins, 2 ties, and 2 losses (6+2+2 = 10 games and 6+2*½ = 7) so this works!
- Fifteen games played by each player in the "lowest," let there be 3 wins, 3 ties, and 9 losses (3+3+9 = 15 games and 3+3*½ = 4.5) so this works!

Thus, 25 is attainable and there are 25 players.

With problems like this, it is always helpful (if given time) to see if your answer is indeed attainable. Checking can be simple as finding whether the solution is extraneous or not to extensive casework like this one. Regardless, I highly recommend you to do so for your own good!

V. Have a Strategy Part II

As you've seen in Part I, strategy is probably the most important key to difficult problems. It can be simple as writing boxes for your work (Art of Problem Solving administrator's tip for organizing your work) to detailed steps.

Now, there are some problems that you look and have a plan set out, and even after all these work, *it doesn't work*. Why did it not work? *I did not make any mistake; my calculations are perfect; I checked my answer and it does not match. WHY WHY WHY!!!* When faced like this scenario, take a step back and view the problem *differently*. Especially with counting problems, it often helps to see if you can solve the problem in more than one way (if you can't though, don't waste time trying to do it). Doing this will clear of any doubt that you had on your work and lead you to correct one. Really, it's important to visualize the problem: don't take it as just black ink and white paper. It is really crucial that you view the problem in a way you would look at a car: rearview, front view, top view, horsepower, color of the car, etc… Remember that you pay more attention to cars like Ferrari and BMW than regular cars. This is same for difficult problem versus simple problems (nevertheless, do not be careless!)

Let's take a look at an example.

(1986 AIME #13) In a sequence of coin tosses one can keep a record of the number of instances when a tail is immediately followed by a h ead, a head is immediately followed by a head, etc. We denote these by TH, HH, etc. For example, in the sequence HHTTHHHHTHHTTTT of 15 coin tosses we observe that there are 5 HH, three HT, two TH and four TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?

Solution:

Where do we start? First of all, which one is first? Head or Tail? This can be answered by doing smaller cases (*investigation*).

HTH - 2 HT and 1 TH

THT - 1 HT and 2 TH

Since we have more TH than HT, let's take a look at this scenario while only focusing on TH and HT (why? Well, it is more difficult to consider TH and HT than HH or TT)

T, H, T, H, T, H, T, H

Why do we like this? It is because we can place additional two H's to do the job for HH and additional five T's to do the job for TT. Let's pause here. Are we doing the right job here?

Well, above sequence is 8 letters and by adding two additional H's and five additional T's, we get 8+2+5 = 15 letters, which is what the problem states!

It is important to stop and see if our reasoning is good. It is ALWAYS necessary to stop and make sure that what you're thinking, writing, or working with works with the problem's requirement.

Now, the answer is product of how many ways to put the two additional H's and to put the five additional T's. Let's take a look at this "sub problem"

- In the ball-and-urn problem, if we have *r* indistinguishable (identical) balls and *n* different boxes, how many ways can we do this? First, place *r* balls… B, B, B, B, …… Now, let's add barriers… B | B, B, | B, B |, … Think about this little bit. Because we want to have *n* different boxes, we need to add *n-1* barriers. And putting *r* balls will takes place *r+n-1* places so: $\frac{(r+n-1)!}{r!(n-1)!}$

  Therefore, let's use the formula.

  The answer then is:

$$\frac{(2+4-1)!}{2!(4-1)!}\times\frac{(5+4-1)!}{5!(4-1)!}=10\times 56=560$$

This concludes the two-part strategy section. I'll add one last one to this section: just remember that what you're thinking matches what you're writing down *and* what the problem is saying. It might seem really obvious but before you plan out anything, make sure that what you're thinking matches with what the problem states because this'll often lead you to right strategy (just like how I emphasized the italicized part) and more importantly, **prevent you from doing wrong problem**. I remember doing on geometry AIME problem, and at the end when I had everything right, I could not get an integer solution. Guess what happened? Wrong number (like.. 81 instead of 18). READ PROBLEM CAREFULLY!

VI. Back to Basics…

I cannot emphasize this enough… Many people often say, "Learn the formulas!" or "Pick up a difficult book and soon, you'll be at the top of your game!" Really, at high school level (exception of USAMO and IMO and TST - Team Selection Test - and MOSP), you do NOT need to learn advanced stuffs like a college student would for Putnam.

That is why… This lesson is for people to remember that you're never too old to go back to BASICS.. And not rely on the formulas! Remember, there is no point in knowing formulas if you don't know how, why, and when to use them!

(1984 AIME #13) Find the value of $10\cot(\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\cot^{-1}21)$

Solution:

When I first looked at this problem, I thought… how the heck am I going to solve this? Many people say this kind of thing when faced with tough problem. My advice is: stick to basics.

First, because we have cotangents added, let's find cotangent addition formula. Second, note that $\cot(\cot^{-1} x) = x$.

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\cot(A+B)=\frac{1}{\tan(A+B)}=\frac{1-\tan A\tan B}{\tan A+\tan B}$$

$$\frac{1-\tan A\tan B}{\tan A+\tan B}=\frac{1-\frac{1}{\cot A}\frac{1}{\cot B}}{\frac{1}{\cot A}+\frac{1}{\cot B}}=\frac{\cot A\cot B-1}{\cot A+\cot B}$$

Now, let A = $\cot^{-1}$ 3, B = $\cot^{-1}$ 7, C = $\cot^{-1}$ 13, and D = $\cot^{-1}$ 21. Therefore, using these notations:

$$\cot(A+B+C+D)=\frac{\cot(A+B)\cot(C+D)-1}{\cot(A+B)+\cot(C+D)}$$

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot A+\cot B}=\frac{3\times 7-1}{3+7}=2$$

$$\cot(C+D)=\frac{\cot C\cot D-1}{\cot C+\cot D}=\frac{13\times 21-1}{13+21}=8$$

$$\cot(A+B+C+D)=\frac{2\times 8-1}{2+8}=\frac{15}{10}$$

So, the final answer is 10 times of that, which is 15.

REMARK: I did not get to take 2008 AIME I this year (due to snow day so my AMC was cancelled) but interestingly, this almost-exact problem was used in 2008 AIME I #8:

$\tan^{-1} 1/3 + \tan^{-1} 1/4 + \tan^{-1} 1/5 + \tan^{-1} 1/n = \pi/4$

So, yes, going back to basics is really important!!!

VII. Final Comment

First of all, thank you for all registering to my competition and hope you guys enjoyed this article. My final advice to anyone participating in math competitions is to enjoy doing it. It’s like playing a videogame: the more you are interested in beating the boss and reaching the next level, the better you’ll get at playing that game. Math competition, after all, is a game of numbers. Books and classes do help but most importantly, you have to have fun doing it.